

GAUSSIAN COMPLEX ZEROS ON THE HOLE EVENT: THE EMERGENCE OF A FORBIDDEN REGION

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ABSTRACT. Consider the Gaussian Entire Function

$$F_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}, \quad z \in \mathbb{C},$$

where $\{\xi_k\}$ is a sequence of independent standard complex Gaussians. This random Taylor series is distinguished by the invariance of its zero set with respect to the isometries of the plane \mathbb{C} . It has been of considerable interest to study the statistical properties of the zero set, particularly in comparison to other planar point processes.

We show that the law of the zero set, conditioned on the function $F_{\mathbb{C}}$ having no zeros in a disk of radius r , and normalized by r^2 , converges to an explicit limiting Radon measure on \mathbb{C} , as $r \rightarrow \infty$. A remarkable feature of this limiting measure is the existence of a large “forbidden region” between a singular part supported on the boundary of the (scaled) hole and the equilibrium measure far from the hole. In particular, this answers a question posed by Nazarov and Sodin, and is in stark contrast to the corresponding result of Jancovici, Lebowitz, and Manificat in the random matrix setting: there is no such region for the Ginibre ensemble.

1. INTRODUCTION

In recent years, particle systems (also known as point processes) involving local repulsion have attracted a lot of attention ([BoBL1, Le, NaS1, PeV, Sos, Wi1, Wi2], to provide a partial list). Two of the most significant mathematical models of translation invariant planar point processes embodying local repulsion are the Ginibre ensemble and the zeros of the standard Gaussian Entire Function (GEF). Both of these processes originate in physics. The Ginibre ensemble was introduced by J. Ginibre ([Gi]), as a non-Hermitian Gaussian matrix model; it also turns out to be the 2D Coulomb gas at a specific temperature. The GEF was introduced by E. Bogomolny, O. Bohigas, and P. Leboeuf ([BoBL1, BoBL2]), in the form of Weyl polynomials. These two ensembles share many similar properties. For instance, their correlations decay as $\exp(-c \cdot \text{distance}^2)$ (see [HoKPV, NaS1, NaS3]).

For a point process, one quantity of interest is the decay rate of the *hole probability*, that is, the probability that a disk of radius r contains no points, as $r \rightarrow \infty$. One can consider this quantity as a rough measure of the mutual repulsion (or “rigidity”) in the process (see [HoKPV, Section 7.2]). Both for the Ginibre ensemble ([JaLM, Sh]), and for the GEF zero process ([Ni1, SoT2]) the hole probability decays like $\exp(-cr^4(1+o(1)))$ (for the Poisson point process, which exhibits no rigidity, the decay rate is $\exp(-cr^2)$). A natural problem that arises is how to describe the behavior of the point process conditioned to have such a large hole. Progress on this problem will allow us to describe the typical configurations that produce this rare event.

Among the main results of this paper is a description of these configurations for the zero set of the GEF, conditioned upon the hole event. We show that beyond a singular component on the boundary of the hole, there is a second “forbidden region” $\{r < |z| < \sqrt{er}\}$, in which the density of the zeros

is negligibly small. This phenomenon is rather surprising, and to the best of our knowledge, this is, in fact, the first example where such a forbidden region in particle systems has been rigorously established, or even heuristically understood. The work of Jancovici, Lebowitz, and Manificat [JaLM] treats in particular the case of the Ginibre ensemble. It shows that conditioning on the hole event, also leads to the formation of a singular component on the boundary of the hole. However, in this case there are no macroscopic restrictions outside the hole (see also [MaNSV], and Section 8 for a short discussion of the one-dimensional case). Figures 1.1 and 1.2 present a simulation of the hole event (with $r = 13$) for the GEF and the Ginibre ensemble, respectively. For more details about this simulation in Section 8.

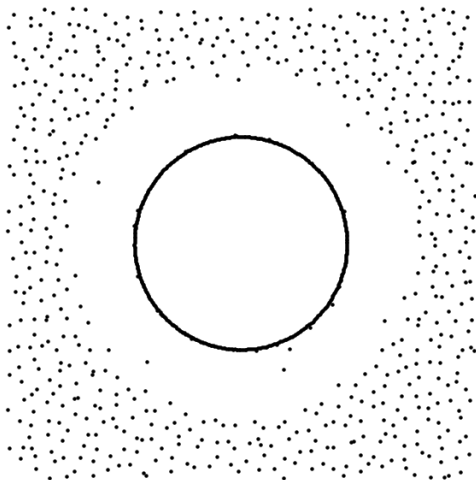


Figure 1.1 - Zeros of GEF on hole event
The black “circle” are zeros of the GEF

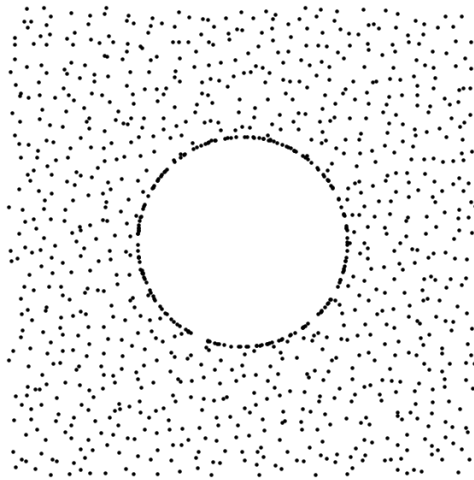


Figure 1.2 - Ginibre ensemble on hole event
See Section 8 for details

Our result for the hole is proved in a more general setting. Given $p \geq 0$, $p \neq 1$, we will condition on the event that the number of zeros in the disk $\{|z| < r\}$ equals $\lfloor pr^2 \rfloor$, and study in details the conditional distribution of the zeros as $r \rightarrow \infty$. The case $p = 0$ corresponds to the hole, $p < 1$ corresponds to a “deficit” of zeros, while $p > 1$ corresponds to an “abundance” of zeros (so called “overcrowding”). To avoid unnecessary long preliminaries, here we will only bring a special case of our results pertaining to the case $p = 0$.

For a compactly supported test-function φ , we put

$$n_{F_C}(\varphi; r) = \sum_{z \in \mathcal{Z}} \varphi\left(\frac{z}{r}\right),$$

where \mathcal{Z} is the random zero set of the GEF. The random variables $n_{F_C}(\varphi; r)$ are called *linear statistics*. In the special case where φ is the indicator function of the unit disk, the corresponding linear statistics is the radial zero counting function. The classical Edelman-Kostlan formula ([HoKPV, Section 2.4]) gives the expected value

$$\mathbb{E}[n_{F_C}(\varphi; r)] = \int_{\mathbb{C}} \varphi\left(\frac{w}{r}\right) \frac{dm(w)}{\pi} = r^2 \cdot \int_{\mathbb{C}} \varphi(w) \frac{dm(w)}{\pi},$$

where m is the Lebesgue measure on \mathbb{C} . We also put

$$d\mu_{Z_0}^{\mathbb{C}}(z) = e \cdot dm_{\{|z|=1\}} + \mathbf{1}_{\{|w| \geq \sqrt{e}\}}(z) \cdot \frac{dm(z)}{\pi},$$

where $m_{\{|z|=1\}}$ is the Lebesgue measure on the unit circle normalized to be a probability measure. Notice that the mass e of the singular component corresponds to the (normalized) area of the disk $\{|z| \leq \sqrt{e}\}$.

Let H_r denote the hole event, when there are no zeros of $F_{\mathbb{C}}$ in the disk $\{|z| < r\}$. By $\mathbb{E}_{H_r}[\cdot]$ (resp. $\mathbb{P}_{H_r}[\cdot]$) we denote the conditional expectation (resp. probability) on H_r . Our main result is the following

Theorem 1. *Fix $\varphi \in C_c^2(\mathbb{C})$ a twice continuously differentiable test function with compact support. As $r \rightarrow \infty$,*

$$\mathbb{E}_{H_r}[n_{F_{\mathbb{C}}}(\varphi; r)] = r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_0}^{\mathbb{C}}(w) + O(r \log^2 r).$$

Let us write $[\mathcal{Z}]$ for the (random) counting measure of \mathcal{Z} . In addition, by \mathcal{Z}_r we denote the zero set conditioned on H_r , and write $[\mathcal{Z}_r]$ for the corresponding counting measure. Recall the space $\mathcal{R}(\mathbb{C})$ of Radon (positive, locally finite) measures on \mathbb{C} can be endowed with the vague topology (such that the mapping $(\nu, \phi) \mapsto \int \phi d\nu$ is continuous). The following corollary of Theorem 1 describes the limiting distribution of the zeros (with appropriate scaling), conditioned on the hole event for large r .

Corollary 1. *As $r \rightarrow \infty$, the scaled conditional zero counting measure $\frac{1}{r^2}[\mathcal{Z}_r](\frac{\cdot}{r}) \rightarrow \mu_{Z_0}^{\mathbb{C}}$ in distribution, where the convergence is in the vague topology.*

Let $n_{F_{\mathbb{C}}}(G) = [\mathcal{Z}](G)$ be the number of zeros of the GEF inside a domain $G \subset \mathbb{C}$. Our second result gives a quantitative upper bound for the actual number of zeros in the forbidden annulus, where the limiting conditional expectation $d\mu_{Z_0}^{\mathbb{C}}$ vanishes.

Theorem 2. *Suppose r is sufficiently large, $\varepsilon \in (r^{-2}, 1)$, that $\gamma \in \left(1 + \frac{1}{2} \log \frac{1}{\varepsilon} (\log r)^{-1}, 2\right]$, and consider the annulus*

$$A_{r,\varepsilon} = \{z \in \mathbb{C} : r(1 + \varepsilon) \leq |z| \leq \sqrt{e}r(1 - \varepsilon)\}.$$

We have

$$\mathbb{P}_{H_r}[n_{F_{\mathbb{C}}}(A_{r,\varepsilon}) \geq r^\gamma] \leq \exp(-C\varepsilon r^{2\gamma}),$$

where $C > 0$ is some numerical constant.

In fact we can prove deviation bounds of similar nature for general smooth linear statistics $\varphi \in C_0^2(\mathbb{C})$, see Theorem 3, Section 2 for the special case of conditioning on the hole event, and Theorem 8, Section 7 for the general setting.

Remark 1. In particular, Theorem 2 implies that for every fixed $\varepsilon, \delta > 0$ we have

$$\mathbb{E}_{H_r}[n_{F_{\mathbb{C}}}(A_{r,\varepsilon})] = O(r^{1+\delta}).$$

This should be compared with the (unconditional) expected number of zeros

$$\mathbb{E}[n_{F_{\mathbb{C}}}(A_{r,\varepsilon})] = (e - 1)r^2 + O(\varepsilon r^2).$$

Our approach is based on precise estimates for the zero set of the GEF via polynomial approximations. We obtain effective deviation bounds for linear statistics of the zeros, which are inspired by a large deviation principle (LDP) for zeros of Gaussian random polynomials due to Zeitouni and Zelditch ([ZeZ], similar LDPs were previously obtained for eigenvalues of random matrices in [BeG, BeZ, HiP]). This approach enables us to reduce a problem on the distribution of the zeros to the solution of a constrained optimization problem in the space of probability measures. We develop a more precise

version of the LDP, which allows us to make the transition from polynomials to entire functions (using results from complex analysis about the variation of the zeros of an analytic function under analytic perturbations). In addition, this allows us to control the error terms, leading in particular to Theorem 2. A key difficulty that arises in this program is that the constrained optimization problem involves a non-standard, non-differentiable functional, which requires the application of potential theoretic techniques.

As mentioned, the techniques of this paper can be effectively used to study other properties of the GEF. We defer the statements of these results to Section 4 and the discussion in Section 8.

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Notation and general remarks. The letters C and c denote positive numerical constants, that do not depend on r and on p . The values of these constants are not essential to the proof, and their value may vary from line to line, or even within the same line. We denote by A, B, C_1, C_2 , etc. constants that we keep fixed throughout the proof in which they appear.

We write $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \{1, 2, 3, \dots\}$. For a finite set J , we sometimes use $\#J$ to denote the size of set (number of elements). We denote by $D(a, r)$ the open disk $\{z \in \mathbb{C} : |z - a| < r\}$. The letter D stands for the unit disk $D(0, 1)$. $[x]$ is the integer part of a number $x \in \mathbb{R}$. For $a, b \in \mathbb{R}$, we write $a \vee b$ ($a \wedge b$) for the maximum (minimum) of the two.

Let $f(r), g(r)$ be positive functions. The notation $g(r) = O(f(r))$ means there is a constant $C = C(p)$ such that $g(r) \leq C f(r)$ for r sufficiently large (possibly depending on p and other parameters). The notation $g(r) = o(f(r))$ means $\frac{g(r)}{f(r)} \xrightarrow{r \rightarrow \infty} 0$.

A function $\varphi : \mathbb{C} \mapsto \mathbb{R}$ admits $\omega(t) : [0, \infty) \mapsto [0, \infty)$ as its modulus of continuity, if for all $z, w \in \mathbb{C}$,

$$|\varphi(z) - \varphi(w)| \leq \omega(|z - w|).$$

By writing $\omega(\varphi; t)$ we mean a function which is admitted as the modulus of continuity of φ . For example, φ is Hölder continuous if $\omega(\varphi; t) = C_\varphi t^\alpha$, for some $C_\varphi > 0, \alpha \in (0, 1]$.

We write m for the Lebesgue measure on \mathbb{C} , while $m_{|z-w|=t}$ is the Lebesgue measure on the circle $|z - w| = t$, normalized to have mass 1.

Analytic functions. Let f be an entire function. We use the standard notation

$$M_f(r) = \max \{|f(z)| : |z| \leq r\}, \quad r \geq 0.$$

We write $\mathcal{Z}(f) = \{z \in \mathbb{C} : f(z) = 0\}$ for the collection of zeros of f (zero set). In principle, multiple zeros appear as many times as their multiplicity (but in this paper all zeros are simple). We denote by $[\mathcal{Z}(f)]$ the counting measure of the zeros of f , that is for a domain G :

$$[\mathcal{Z}(f)](G) = \#\{z \in G : f(z) = 0\}.$$

We write $n_f(r)$ for the number of zeros of the function f inside the closed disk $\overline{D(0, r)} = \{|z| \leq r\}$. For the GEF $F_{\mathbb{C}}$ we usually just write $M(r) = M_{F_{\mathbb{C}}}(r)$ and $n(r) = n_{F_{\mathbb{C}}}(r)$ (these are random variables).

Let φ be a test function with compact support, the linear statistics of f with respect to (w.r.t.) φ is given by

$$n_f(\varphi; r) = \sum_{z \in \mathcal{Z}(f)} \varphi\left(\frac{z}{r}\right), \quad r > 0.$$

Probability and negligible events. We denote events by E, F , etc., by E^c the complement of the event E , and by $\bigsqcup E_k$ the disjoint union of the events E_k . We write $\mathbb{P}[E]$ for the probability of the event E (the probability space will always be clear from the context). An event $E = E(r)$ will be called *negligible* with respect to $F(p; r) = \{n(r) \leq pr^2\}$ ($M(p; r) = \{n(r) \geq pr^2\}$) if $\mathbb{P}[E] = o(\mathbb{P}[F(p; r)])$ (respectively, $\mathbb{P}[E] = o(\mathbb{P}[M(p; r)])$). In that case we have $\mathbb{P}[F(p; r)] \leq \mathbb{P}[F(p; r) \cap E^c] + \mathbb{P}[E] = (1 + o(1)) \mathbb{P}[F(p; r) \cap E^c]$.

If X is a random variable, then $\mathbb{E}[X]$ is its mean (expected value), and $\text{Var}[X]$ is its variance (if they exist). We write $X|_F$ to denote the random variable X conditioned on the event F . We have $\mathbb{E}_F[X] = (\mathbb{P}[F])^{-1} \cdot \mathbb{E}[X \cdot \mathbf{1}_F(\cdot)]$, where $\mathbf{1}_F(\cdot)$ is the indicator random variable of the event F .

Measures. We consider mainly probability measures on the complex plane, which we denote by $\mathcal{M}_1(\mathbb{C})$. Sometimes we consider Radon (locally finite) measures. All the measures we work with are assumed to be Borel measures. All sets are assumed to be Borel measurable. We denote collections (or sets) of measures by \mathcal{C}, \mathcal{D} , etc.

Potential theory. Let $\mu \in \mathcal{M}_1(\mathbb{C})$. We write

$$U_\mu(z) = \int_{\mathbb{C}} \log|z - w| \, d\mu(w), \quad \Sigma(\mu) = \int_{\mathbb{C}} \int_{\mathbb{C}} \log|z - w| \, d\mu(z) \, d\mu(w) = \int_{\mathbb{C}} U_\mu(z) \, d\mu(z),$$

for the logarithmic potential and the logarithmic energy of the measure, respectively. A measure is said to have finite logarithmic energy if $|\Sigma(\mu)| < \infty$. Sometime we use the same notation for the logarithmic potential and energy of signed measures (with finite total variation). For more details, see Appendix B.

2. IDEA OF THE PROOF

Recall the GEF is the Gaussian Entire function, given by the Taylor series

$$(2.1) \quad F_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}, \quad z \in \mathbb{C},$$

where $\{\xi_k\}$ is a sequence of independent standard complex Gaussians (i.e., the density of ξ_k with respect to m , the Lebesgue measure on the complex plane \mathbb{C} , is $\frac{1}{\pi} e^{-|z|^2}$). The hole event at radius r , denoted H_r , is the event where $F_{\mathbb{C}}$ has no zeros inside the disk $D(0, r) = \{|z| < r\}$. Suppose $\varphi \in C_0^2(\mathbb{C})$ is a test function, which is twice continuously differentiable and with compact support, and define

$$\mathfrak{D}(\varphi) = \|\nabla \varphi\|_{L^2(m)}^2 = \int_{\mathbb{C}} (\varphi_x^2 + \varphi_y^2) \, dm(z).$$

Let $n_{F_{\mathbb{C}}}(\varphi; r)$ be the linear statistics associated with φ , we would like to consider an event where the linear statistics is far from the conditional limiting measure $\mu_{Z_0}^{\mathbb{C}}$. Theorems 1 and 2 will be deduced from the following (conditional) deviation inequality.

Theorem 3. Suppose $C' > 0$ is fixed, and that r is sufficiently large. For $\lambda \in (0, C'r^2)$ we have,

$$\mathbb{P}_{H_r} \left[\left| n_{F_{\mathbb{C}}}(\varphi; r) - r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_0}^{\mathbb{C}}(w) \right| \geq \lambda \right] \leq \exp \left(-\frac{C}{\mathfrak{D}(\varphi)} \cdot \lambda^2 + C_{\varphi} r^2 \log^2 r \right),$$

where $C > 0$ is a numerical constant and $C_{\varphi} > 0$ is a constant depending only on φ .

A key ingredient in our proof is the fact that the zero set of the random polynomial, obtained by truncating the Taylor series (2.1) at a large degree N (which depends on r), serves as a good approximation for the zeros set of $F_{\mathbb{C}}$. The advantage of working with the polynomial is that one can write down a closed form expression for the joint density of its zeros. We can then identify any given instance of such a random zero set with the corresponding empirical measure (i.e., the probability measure with equal weights on the zeros). Another key ingredient is that at the exponential scale, this joint density can be further approximated by a certain functional that acts on the empirical measure. For the problem at hand, we can focus our attention on the behavior of this functional on an appropriate subset of empirical measures, namely those that put no mass on the “hole”.

2.1. Truncation of the Taylor series. Suppose $r > 0$ is large. Let φ be a test function supported on the disk $D(0, B)$, where $B \geq 1$ is fixed. Let $N \in \mathbb{N}$ be a large parameter (depending on r). We also introduce the large parameter $L = r + O(\frac{1}{r})$. We will work with the scaled polynomials

$$(2.2) \quad P_{N,L}(z) = \sum_{k=0}^N \xi_k \frac{(Lz)^k}{\sqrt{k!}}, \quad z \in \mathbb{C}.$$

Roughly speaking, L would correspond to the size of the hole, and N is the degree of polynomial truncation of $F_{\mathbb{C}}$. As a matter of fact, we will choose N such that $\alpha \stackrel{\text{def}}{=} Nr^{-2}$ is of order $\log r$. In what follows, we denote by E_{reg} an event for which the (scaled) zeros of the polynomial $P_{N,L}$ serve as a good approximation for the zeros of $F_{\mathbb{C}}$ inside the disk $D(0, Br)$. We can choose E_{reg} so that $\mathbb{P}[E_{\text{reg}}^c] \leq \exp(-Ar^4)$, for some large constant $A > 0$. For the details see Subsection 4.1.

2.2. The joint distribution of the zeros of $P_{N,L}$. Put $d\mu_L(w) = \frac{L^2}{\pi} e^{-L^2|w|^2} dm(w)$, where m is Lebesgue measure on \mathbb{C} (μ_L is a probability measure). Let us denote by z_1, \dots, z_N the zeros of the polynomial $P_{N,L}$, and in addition write $\underline{z} = (z_1, \dots, z_N)$. Lemma 11 (in Appendix A) shows that the joint probability density of the zeros (in uniform random order), with respect to Lebesgue measure on \mathbb{C}^N , is given by

$$f(\underline{z}) = f(z_1, \dots, z_N) = A_L^N |\Delta(\underline{z})|^2 \left(\int_{\mathbb{C}} \prod_{j=1}^N |q_{\underline{z}}(w)|^2 d\mu_L(w) \right)^{-(N+1)},$$

where

$$q_{\underline{z}}(w) = \prod_{j=1}^N (w - z_j),$$

$|\Delta(\underline{z})|^2 = \prod_{j \neq k} |z_j - z_k|$, and A_L^N is the normalization constant.

For a probability measure $\mu \in \mathcal{M}_1(\mathbb{C})$ we denote by $U_{\mu}(z)$, $\Sigma(\mu)$ its logarithmic potential and logarithmic energy, respectively (for the definitions we refer to the notation section of the introduction). Let $\mu_{\underline{z}} = \frac{1}{N} \sum_{j=1}^N \delta_{z_j}$ be the empirical probability measure of the zeros. Instead of working with the squared Vandermonde $|\Delta(\underline{z})|^2$, we would like to work with the logarithmic energy functional $\Sigma(\mu)$.

However, the latter is not well-defined for discrete measures. Thus, it will be required to introduce the smoothed empirical measure $\mu_z^t = \mu_z \star m_{\{|z|=t\}}$, where $t = t(r) > 0$ is a small parameter.

Consider the functional $I_\alpha : \mathcal{M}_1(\mathbb{C}) \rightarrow [0, \infty]$ given by

$$I_\alpha(\nu) = 2 \sup_{w \in \mathbb{C}} \left\{ U_\nu(w) - \frac{|w|^2}{2\alpha} \right\} - \Sigma(\nu).$$

The uniform probability measure on the disk $D(0, \sqrt{\alpha})$, denoted by μ_α , is known to be the unique global minimizer of I_α . In Subsection (4.2) we show one can bound $f(z)$ by

$$\exp \left(-N^2 \left[I_\alpha(\mu_z^t) - I_\alpha(\mu_\alpha) + o(1) \right] \right).$$

Let $Z \subset \mathbb{C}^N$ be a nice subset of possible ‘configurations’ of the zeros. Roughly speaking, for such Z , we show that

$$(2.3) \quad \mathbb{P}[\{z \in Z\} \cap E_{\text{reg}}] = \int_{Z \cap E_{\text{reg}}} f(z) \, dm(z) \leq \exp \left(-N^2 \left[\inf_{z \in Z} I_\alpha(\mu_z^t) - I_\alpha(\mu_\alpha) + o(1) \right] \right).$$

For the reader who is acquainted with the theory of large deviations, this upper bound is similar in spirit to the large deviations upper bound for empirical measures of random polynomials obtained in [ZeZ].

2.3. Conditioning on the hole event and a constrained optimization problem. On the hole event H_r , there are no zeros of $P_{N,L}$ inside the disk $\{|z| \leq (1 - \delta) \frac{r}{L}\}$ (for a small $\delta > 0$, depending on r). The factor $1 - \delta$ appears as a side-effect of the truncation of $F_{\mathbb{C}}$. The factor L is the result of the scaling of the zeros. Choosing the parameter L slightly smaller than r , we see that $\mu_z^t(D) = 0$ for $t > 0$ sufficiently small (D is the unit disk). The upper bound (2.3) suggests that in order to bound the probability of the hole event we should find the minimizer of I_α over the set

$$\mathcal{H} = \{\nu \in \mathcal{M}_1(\mathbb{C}) : \nu(D) = 0\}.$$

Since the functional I_α is lower semi-continuous and strictly convex, this minimizer exists and is unique. If the value of this minimizer happens to agree with the lower bound for the hole probability, we can consider this minimizing measure to be (an appropriately scaled limit of) the empirical measure of the most likely configuration of zeros that gives rise to the hole event H_r .

To be somewhat more precise, we set $t = r^{-C_2}$, for some constant $C_2 \geq 4$. Let us first obtain an upper bound for the probability of the hole event. As suggested by the discussion above, we want to consider the configurations in

$$Z = \{z \in \mathbb{C}^N : \mu_z^t(D) = 0\}.$$

Since $\{\mu_z^t : z \in Z\} \subset \mathcal{H}$, using (2.3) we obtain the bound

$$\mathbb{P}[H_r \cap E_{\text{reg}}] \leq \exp \left(-N^2 \left[\inf_{z \in \mathcal{H}} I_\alpha(\mu_z^t) - I_\alpha(\mu_\alpha) + o(1) \right] \right).$$

In Section 5 we find that the minimizer of $I_\alpha(\nu)$ over the set \mathcal{H} is given by

$$d\mu_{Z_0}^\alpha(z) = \frac{e}{\alpha} dm_{\{|z|=1\}} + \frac{1}{\alpha} \mathbf{1}_{\{\sqrt{e} \leq |z| \leq \sqrt{\alpha}\}}(z) \cdot \frac{dm(z)}{\pi}.$$

A short calculation shows that

$$(2.4) \quad \exp \left(-N^2 \left[\inf_{\nu \in \mathcal{H}} I_\alpha(\nu) - I_\alpha(\mu_\alpha) \right] \right) = \exp \left(-N^2 [I_\alpha(\mu_{Z_0}^\alpha) - I_\alpha(\mu_\alpha)] \right) = \exp \left(-\frac{e^2}{4} r^4 \right).$$

Since $\mathbb{P}[E_{\text{reg}}^c] \leq \exp(-Ar^4)$, for some large constant $A > 0$, we obtain the required bound for $\mathbb{P}[H_r]$. See Section 4 for a proof (for general $p \geq 0$, $p \neq 1$).

2.4. Large deviations for linear statistics. Now we would like to consider configurations where the empirical measure is ‘far’ from the minimizing measure $\mu_{Z_0}^\alpha$. Recall $N = \alpha r^2$ and that α is a large parameter. It follows that $\mu_{Z_0}^\mathbb{C}(D(0, \sqrt{\alpha})) = \alpha$, and thus

$$(2.5) \quad r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_0}^\mathbb{C}(w) = N \cdot \int_{\mathbb{C}} \varphi(w) d\mu_{Z_0}^\alpha(w).$$

In addition, on the event E_{reg} , we have

$$(2.6) \quad \int_{\mathbb{C}} \varphi(w) d\mu_{\underline{z}}^t(w) = N \cdot n_{F_{\mathbb{C}}}(\varphi; r) (1 + o(1)).$$

Consider now the event

$$L(0, \varphi, \lambda; r) = \left\{ \left| n_{F_{\mathbb{C}}}(\varphi; r) - r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_0}^\mathbb{C}(w) \right| \geq \lambda \right\}.$$

By (2.5) and (2.6), on the event $L(0, \varphi, \lambda; r) \cap H_r \cap E_{\text{reg}}$ we have $\mu_{\underline{z}}^t(D) = 0$, and

$$(2.7) \quad \left| \int_{\mathbb{C}} \varphi(w) d\mu_{\underline{z}}^t(w) - \int_{\mathbb{C}} \varphi(w) d\mu_{Z_0}^\alpha(w) \right| \geq \frac{\lambda}{N} - \text{error terms}.$$

Therefore, we now consider the configurations in

$$Z' = \left\{ \underline{z} \in \mathbb{C}^N : \mu_{\underline{z}}^t(D) = 0 \text{ and } \left| \int_{\mathbb{C}} \varphi(w) d\mu_{\underline{z}}^t(w) - \int_{\mathbb{C}} \varphi(w) d\mu_{Z_0}^\alpha(w) \right| \geq \frac{\lambda}{N} (1 + o(1)) \right\}.$$

In order to obtain a large deviation bound for the linear statistics, we need some estimates for the convexity of the functional I_α . By Claim 11 we have for any measure $\nu \in \mathcal{H}$ which satisfies

$$\left| \int_{\mathbb{C}} \varphi(w) d\nu(w) - \int_{\mathbb{C}} \varphi(w) d\mu_{Z_0}^\alpha(w) \right| \geq x,$$

the following bound

$$(2.8) \quad I_\alpha(\nu) \geq I_\alpha(\mu_{Z_0}^\alpha) + \frac{2\pi}{\mathfrak{D}(\varphi)} \cdot x^2.$$

This implies that for $\underline{z} \in Z'$ we have

$$I_\alpha(\mu_{\underline{z}}^t) \geq I_\alpha(\mu_{Z_0}^\alpha) + \frac{2\pi}{\mathfrak{D}(\varphi)} \cdot \left(\frac{\lambda}{N} \right)^2 - \text{error terms}.$$

Finally, the bound (2.3) (together with (2.4)) gives

$$\begin{aligned} \mathbb{P}[L(0, \varphi, \lambda; r) \cap H_r \cap E_{\text{reg}}] &\leq \mathbb{P}[Z' \cap E_{\text{reg}}] \leq \exp \left(-N^2 \left[\inf_{\underline{z} \in Z'} I_\alpha(\mu_{\underline{z}}^t) - I_\alpha(\mu_\alpha) + o(1) \right] \right) \\ &\leq \exp \left(-N^2 \left[\frac{C}{\mathfrak{D}(\varphi)} \cdot \left(\frac{\lambda}{N} \right)^2 + I_\alpha(\mu_{Z_0}^\alpha) - I_\alpha(\mu_\alpha) \right] + \text{error terms} \right) \\ &\leq \mathbb{P}[H_r] \exp \left(-\frac{C}{\mathfrak{D}(\varphi)} \cdot \lambda^2 + \text{error terms} \right), \end{aligned}$$

hence we obtain Theorem 3. See Subsection 7.3 for the details.

3. PRELIMINARIES

Throughout the paper we will use the following standard bounds for the factorial

$$(3.1) \quad \left(\frac{k}{e}\right)^k \leq k! \leq 3\sqrt{k} \left(\frac{k}{e}\right)^k, \quad k \geq 1.$$

The lower bound follows immediately from the series expansion for e^x , and the upper bound by induction and using the inequality $(1 + \frac{1}{k})^{k+\frac{1}{2}} > e$. Recall that for a standard Gaussian random variable ξ , we have $|\xi|^2 \sim \exp(1)$, i.e. $\mathbb{P}[|\xi| \geq \lambda] = \exp(-\lambda^2)$. In what follows, we frequently use the estimate

$$\mathbb{P}[|\xi| \leq \lambda] \in \left[\frac{\lambda^2}{2}, \lambda^2\right], \quad \lambda < 1.$$

3.1. Estimates for the GEF. When studying the distribution of the zeros of the GEF, it is usually easier to work with a truncation of its Taylor series. We use simple estimates for the ‘tail’ of the series, to control the error that is introduced by the truncation.

Let $\{\xi_k\}_{k=0}^\infty$ be a sequence of i.i.d. standard complex Gaussians. We need the following simple estimates.

Lemma 1. *Let $r > 2$. We have*

$$\mathbb{P}\left[\forall k \ |\xi_k| \leq \sqrt{r^6 + k}\right] \geq 1 - Ce^{-r^6}.$$

Proof. For all $k \geq 0$, we have

$$\mathbb{P}\left[|\xi_k| \leq \sqrt{r^6 + k}\right] = 1 - \exp(-(r^6 + k)).$$

Therefore, using the independence of the ξ_k s

$$\begin{aligned} \mathbb{P}\left[\forall k \ |\xi_k| \leq \sqrt{r^6 + k}\right] &= \prod_{k=0}^{\infty} (1 - \exp(-(r^6 + k))) = \exp\left(\sum_{k=0}^{\infty} \log[1 - \exp(-(r^6 + k))]\right) \\ &\geq \exp\left(-2 \cdot \sum_{k=0}^{\infty} \exp(-(r^6 + k))\right) \geq \exp(-4 \cdot e^{-r^6}) \geq 1 - 4 \cdot e^{-r^6}. \end{aligned}$$

□

Lemma 2. *Let $x \geq 10$. For $N \in \mathbb{N}$,*

$$\mathbb{P}\left[\sum_{k=0}^N |\xi_k|^2 > x(N+1)\right] \leq \exp\left(-\frac{Nx}{2}\right).$$

Proof. Let $t \in (0, 1)$. Using Markov’s inequality and the independence of the ξ_k s, we get

$$\begin{aligned} \mathbb{P}\left[\sum_{k=0}^N |\xi_k|^2 > x(N+1)\right] &= \mathbb{P}\left[\exp\left(t \cdot \sum_{k=0}^N |\xi_k|^2\right) > e^{tx(N+1)}\right] \leq e^{-tx(N+1)} \mathbb{E}\left[\exp\left(t \cdot \sum_{k=0}^N |\xi_k|^2\right)\right] \\ &= e^{-tx(N+1)} \left(\mathbb{E}\left[e^{t|\xi|^2}\right]\right)^{N+1}, \end{aligned}$$

where ξ is a standard complex Gaussian. Using the fact $\mathbb{E} \left[e^{t|\xi|^2} \right] = \frac{1}{1-t}$, and taking $t = \frac{x-1}{x}$, we then have

$$\mathbb{P} \left[\sum_{k=0}^N |\xi_k|^2 > x(N+1) \right] \leq (xe^{1-x})^{N+1} \leq \exp \left(-\frac{1}{2}Nx \right),$$

since $x \geq 10$. \square

For $N \in \mathbb{N}$ define the tail of the GEF to be the series by

$$T_N(z) = \sum_{k=N+1}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}, \quad z \in \mathbb{C}.$$

Lemma 3. *Let $r > 0$ be sufficiently large, $\lambda > 4$, and $B \in \left[0, \frac{\sqrt{\lambda}}{2}\right]$. Outside an exceptional event of probability at most $\exp(-Cr^6)$, we have for any $N \in \mathbb{N}$, such that $N \geq \lambda r^2$,*

$$|T_N(z)| \leq \exp \left(\frac{N}{2} \log \left(\frac{4B^2}{\lambda} \right) \right), \quad |z| \leq Br.$$

Proof. By the previous lemma, after discarding an exceptional event, we may assume $|\xi_k| \leq \sqrt{r^6 + k}$ for all $k \in \mathbb{N}$. Let us write

$$d_k = \sqrt{r^6 + k} \cdot \frac{r^k}{\sqrt{k!}}, \quad k \in \mathbb{N}.$$

Then, for $k \geq N$,

$$\frac{d_{k+1}}{d_k} = \sqrt{1 + \frac{1}{r^6 + k}} \cdot \frac{r}{\sqrt{k+1}} \leq \sqrt{\frac{2}{\lambda}}.$$

Therefore, for $|z| \leq Br$, we have

$$|T_N(z)| \leq \sum_{k=N+1}^{\infty} |\xi_k| \frac{B^k r^k}{\sqrt{k!}} = \sum_{k=N+1}^{\infty} B^k \cdot d_k \leq B^{N+1} d_{N+1} \sum_{k=0}^{\infty} \left(B \sqrt{\frac{2}{\lambda}} \right)^k \leq C \cdot B^{N+1} \cdot d_{N+1}.$$

Using (3.1),

$$d_{N+1} = \sqrt{r^6 + N+1} \cdot \frac{r^{N+1}}{\sqrt{(N+1)!}} \leq C \sqrt{r^6 + N+1} \left(\frac{er^2}{N+1} \right)^{\frac{N+1}{2}} \leq CN^2 \left(\frac{3}{\lambda} \right)^{\frac{N+1}{2}} \leq \left(\frac{7}{2\lambda} \right)^{\frac{N}{2}},$$

since $CN^2 \leq \left(\frac{7}{6} \right)^{\frac{N}{2}}$, for N sufficiently large. Similarly $CB \leq N \leq \left(\frac{8}{7} \right)^{N/2}$ for N sufficiently large, hence the required estimate is obtained. \square

For the upper bound estimate we need an ‘a priori’ bound for the number of zeros of the GEF inside a disk. The following estimate follows immediately from [Kr, Theorem 3].

Corollary 2. *Let $r > 0$ be large enough. We have*

$$\mathbb{P} [n_F(r) \geq r^3] \leq \exp(-r^6).$$

Let $M_{F_C}(\rho) = \max \{|F_C(z)| : |z| \leq \rho\}$. We want to avoid the event where the GEF is very small (in absolute value) inside a large disk. We use the following simple estimate (cf. with the more accurate [Ni1, Lemma 7]).

Lemma 4. *Let $x > 0$. For $\rho > 1$ we have*

$$\mathbb{P} [M_{F_C}(\rho) \leq \exp(-x)] \leq \exp(-2x\rho^2).$$

Proof. Assuming $M_{F_{\mathbb{C}}}(r) \leq \exp(-x) < 1$ and using Cauchy's estimate for the coefficients of $F_{\mathbb{C}}$ we find that

$$|\xi_k| \frac{\rho^k}{\sqrt{k!}} \leq M_{F_{\mathbb{C}}}(\rho) \leq \exp(-x), \quad \forall k \in \mathbb{N}.$$

Notice that the sequence $\frac{\rho^k}{\sqrt{k!}}$ is increasing from $k = 0$ to $k = \lfloor \rho^2 \rfloor$. Therefore, we get

$$|\xi_k| \leq \exp(-x), \quad k \in \{0, \dots, \lfloor \rho^2 \rfloor\},$$

and the probability of this event is at most $\exp(-2x(\lfloor \rho^2 \rfloor + 1)) \leq \exp(-2x\rho^2)$. \square

We also want to control the probability the GEF is too large inside a large disk (this is a very rare event).

Lemma 5 (See [SoT2, Lemma 1]). *For $\rho > 0$ large enough, we have*

$$\mathbb{P}[M_{F_{\mathbb{C}}}(\rho) \geq \exp(\rho^2)] \leq \exp(-\exp(\rho^2)).$$

3.2. Perturbation of Zeros of Analytic Functions. Let f be an entire function and denote by w_1, \dots, w_m the zeros of f in $D(0, r)$ (including multiplicities). For $0 < \gamma \leq \frac{1}{4}$, set

$$C_{\gamma}(r) = \bigcup_{k=1}^m D(w_k, \gamma), \quad E_{\gamma}(r) = D(0, r) \setminus C_{\gamma}, \quad \text{and} \quad m_f(r; \gamma) = \min_{z \in E_{\gamma}} |g(z)|.$$

The following theorem is a restatement of a theorem of Rosenbloom ([Ro, Theorem 4]) for the unit disk. It gives an effective lower bound for the modulus of an analytic function, outside a neighborhood of its zeros.

Theorem 4. *Let f be an entire function. Suppose that $|f(z_0)| \geq 1$ for some $z_0 \in \mathbb{C}$ with $|z_0| = \rho > 0$. Let $0 < r \leq \frac{\rho}{2}$, and suppose that $E_{\gamma}(r) \neq \emptyset$, then*

$$m_f(r; \gamma) \geq \exp\left(-C \log M_f(3\rho) \log \frac{1}{\gamma}\right).$$

We can use the previous theorem to control the perturbation of the zeros of analytic functions, when we add an ‘error term’ of small modulus.

Lemma 6. *Let f, g be entire functions, and $B, \rho \geq 1$. Suppose that f has at most $M > 0$ zeros in the disk $D(0, 2B\rho)$ and let $0 < \gamma < \frac{\rho}{2M}$. In addition, assume $M_g(2B\rho) < m_f(2B\rho; \gamma)$. Then,*

$$n_{f+g}(\rho' - 2M\gamma) \leq n_f(\rho') \leq n_{f+g}(\rho' + 2M\gamma), \quad \forall \rho' \in (2M\gamma, 2B\rho - 2M\gamma).$$

Furthermore, if φ is a test function supported on $D(0, B)$, with modulus of continuity $\omega(\varphi; t)$, then

$$|n_f(\varphi; \rho) - n_{f+g}(\varphi; \rho)| \leq M \cdot \omega\left(\varphi; \frac{2M\gamma}{\rho}\right).$$

Proof. Let $C_{\gamma} = C_{\gamma}(2B\rho)$. We can write $C_{\gamma} = \bigcup C_j$ where C_j are the connected components of C_{γ} (tangent disks are not connected). Notice that the diameter of each component is at most $2\gamma \cdot M$, and that by Rouché's theorem the number of zeros of f and $f + g$ is the same in each component. Therefore, we find that

$$\begin{aligned} n_f(\rho') &\leq \#\{\text{zeros of } f \text{ in components intersecting } |z| \leq \rho'\} \\ &= \#\{\text{zeros of } f + g \text{ in components intersecting } |z| \leq \rho'\} \\ &\leq n_{f+g}(\rho' + 2M\gamma). \end{aligned}$$

The lower bound is obtained in the same way. Similarly, if $w, w' \in C_j$ are two points in the same component C_j , then

$$\left| \varphi\left(\frac{w}{\rho}\right) - \varphi\left(\frac{w'}{\rho}\right) \right| \leq \omega\left(\varphi; \frac{2M\gamma}{\rho}\right).$$

We conclude that

$$|n_f(\varphi; \rho) - n_{f+g}(\varphi; \rho)| = \left| \sum_{z \in \mathcal{Z}(f)} \varphi\left(\frac{z}{\rho}\right) - \sum_{z' \in \mathcal{Z}(f+g)} \varphi\left(\frac{z'}{\rho}\right) \right| \leq M \cdot \omega\left(\varphi; \frac{2M\gamma}{\rho}\right).$$

□

Remark 2. If f has no zeros in the disk $D(0, 2B\rho)$, then, under the assumptions of the lemma, $f + g$ also has no zeros there. Thus, the results of the lemma follow also in this case.

3.3. Truncation of the GEF. We now explain how to truncate the power series $F_{\mathbb{C}}$, such that we can obtain a polynomial whose zeros (inside a disk $D(0, Cr)$) are very close to the zeros of $F_{\mathbb{C}}$. This introduces some technical complications. Let $N \in \mathbb{N}$. We would like to split the GEF in the following way,

$$F_{\mathbb{C}}(z) = \sum_{k=0}^N \xi_k \frac{z^k}{\sqrt{k!}} + \sum_{k=N+1}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}} \stackrel{\text{def}}{=} P_N(z) + T_N(z), \quad z \in \mathbb{C}.$$

We now define a ‘regular’ event, on which the GEF has desirable properties. The complement of this event is negligible. An additional technical issue is the fact we need control over the leading coefficient of P_N , that is over $|\xi_N|$ (we want to keep it not too small). This means that in order to make the exceptional set small, we have to pick a random value for N . Recall that $n_{F_{\mathbb{C}}}(r)$ is the number of zeros of $F_{\mathbb{C}}$ in the disk $\{|z| \leq r\}$, and that

$$M_{F_{\mathbb{C}}}(r) = \sup_{|z| \leq r} |F_{\mathbb{C}}(z)|.$$

Lemma 7. *Let $\rho > 0$ be sufficiently large, $A \geq 1$, $\lambda > 16$, and $B \in \left[1, \frac{\sqrt{\lambda}}{2}\right]$. There exist an event E_{reg} with the following properties:*

- (1) $\mathbb{P}[E_{\text{reg}}^c] \leq \exp(-C \cdot AB^4 \rho^4) + \exp(-C\lambda \rho^4)$.
- (2) *On the event E_{reg} we have:*
 - (a) *For any $N \in \mathbb{N}$ with $N \geq \lambda \rho^2$ we have $|T_N(z)| \leq \exp\left(\frac{N}{2} \log\left(\frac{16B^2}{\lambda}\right)\right)$, $\forall |z| \leq 2B\rho$.*
 - (b) $n_{F_{\mathbb{C}}}(2B\rho) \leq (2B\rho)^3$.
 - (c) $M_{F_{\mathbb{C}}}(6B\rho) \leq \exp(36 \cdot B^2 \rho^2)$.
 - (d) $M_{F_{\mathbb{C}}}(4B\rho) \geq \exp(-AB^2 \rho^2)$.
- (3) *Let $N_0 = \lfloor \lambda \rho^2 \rfloor + 1$, $N_1 = \lfloor 2\lambda \rho^2 \rfloor + 1$. We have $E_{\text{reg}} = \biguplus_{N=N_0}^{N_1} E_{\text{reg}}^N$, where on the event E_{reg}^N we have $|\xi_N| \geq \exp(-\rho^2)$.*
- (4) *For any $N \in \{N_0, \dots, N_1\}$ we have $\sum_{k=0}^N |\xi_k|^2 \leq C\lambda \rho^4$.*

Proof. The properties in (2) follow by combining the statements of Lemma 3, Corollary 2, Lemma 4, and Lemma 5. The probability of the exceptional event in Lemma 4 is the largest one.

Since the ξ_k are independent, the probability that $|\xi_k| < \exp(-\rho^2)$ for all $k \in \{N_0, \dots, N_1\}$ is at most $\exp(-C(N_1 - N_0 + 1)\rho^2) \leq \exp(-C\lambda \rho^4)$. To make the events E_{reg}^N disjoint (this is not essential for our estimates), we can choose N to be smallest value of $k \in \{N_0, \dots, N_1\}$ such that $|\xi_k| \geq \exp(-\rho^2)$.

Finally, by Lemma 2, we have

$$\mathbb{P} \left[\sum_{k=0}^{N_1} |\xi_k| > 3\lambda\rho^4 \right] \leq \mathbb{P} \left[\sum_{k=0}^{N_1} |\xi_k| > \rho^2 (N_1 + 1) \right] \leq \exp \left(-\frac{N_1}{2} \cdot \rho^2 \right) \leq \exp (-C\lambda\rho^4).$$

□

We would now like to apply Lemma 6 to show that we can approximate the zeros of the GEF.

Lemma 8. *Let ρ, A, λ, B be as above. In addition, let $\gamma > 0$ and $M_0 = 8B^3\rho^3$. Suppose $\gamma \leq \frac{\rho}{2M_0}$, $\lambda = o(\rho)$ and that $B^2 A \log \frac{1}{\gamma} = o(\lambda \log(\frac{\lambda}{16B^2}))$ is satisfied. Then, on the event E_{reg} , we have*

$$m_{F_{\mathbb{C}}}(2B\rho; \gamma) > M_{T_N}(2B\rho), \quad \forall N \in \{N_0, \dots, N_1\},$$

and on E_{reg}^N we also have

$$\frac{1}{|\xi_N|} \sum_{k=0}^N |\xi_k|^2 \leq \exp(C\rho^2).$$

In particular, this implies,

$$n_{P_N}(\rho - 2M_0\gamma) \leq n_{F_{\mathbb{C}}}(\rho) \leq n_{P_N}(\rho + 2M_0\gamma).$$

Furthermore, if φ is a test function supported on $D(0, B)$, with modulus of continuity $\omega(\varphi; t)$, then

$$|n_{F_{\mathbb{C}}}(\varphi; \rho) - n_{P_N}(\varphi; \rho)| \leq CM_0 \cdot \omega\left(\varphi; \frac{2M_0\gamma}{\rho}\right).$$

Proof. The bound for $\frac{1}{|\xi_N|} \sum_{k=0}^N |\xi_k|^2$ follows immediately from the previous lemma. Let $N \in \{N_0, \dots, N_1\}$. On E_{reg} we have the bound

$$|T_N(z)| \leq \exp\left(\frac{N}{2} \log\left(\frac{16B^2}{\lambda}\right)\right), \quad \forall |z| \leq 2B\rho.$$

According to Property (2d) in the previous lemma, and the maximum modulus principle, there exists a point z_0 , with $|z_0| = 4B\rho$ and $|F_{\mathbb{C}}(z_0)| \geq \exp(-AB^2\rho^2)$. We now set

$$\tilde{F}(z) = \frac{F_{\mathbb{C}}(z)}{F_{\mathbb{C}}(z_0)}, \quad \tilde{M}(t) = \sup \left\{ |\tilde{F}(z)| : |z| \leq t \right\}.$$

Thus, using Property (2c), we find that $\log \tilde{M}(6B\rho) \leq CB^2\rho^2 + AB^2\rho^2$. Applying Theorem 4 to the function \tilde{F} , we find that

$$m_{F_{\mathbb{C}}}(2B\rho; \gamma) \geq \exp\left(- (CB^2\rho^2 + AB^2\rho^2) \log \frac{1}{\gamma}\right) \geq \exp\left(-CB^2\rho^2 \cdot A \log \frac{1}{\gamma}\right),$$

where we used the fact that $|F_{\mathbb{C}}(z)| \geq |\tilde{F}(z)| \exp(-AB^2\rho^2)$. In order to obtain $m_{F_{\mathbb{C}}}(2B\rho; \gamma) > |T_N(z)|$ for all $z \in D(0, 2B\rho)$, we should have

$$\frac{N}{2} \log\left(\frac{16B^2}{\lambda}\right) < -CB^2\rho^2 \cdot A \log \frac{1}{\gamma},$$

which is satisfied by our requirements on λ and γ (recall N is of order $\lambda\rho^2$). Finally, by Property 2b, we have that $M \stackrel{\text{def}}{=} n(2B\rho) \leq 8B^3\rho^3 = M_0$ (w.l.o.g. we may assume that $M > 0$, see the remark after Lemma 6). Lemma 6, applied to the functions $F_{\mathbb{C}}$ and $-T_N$, then implies,

$$n_{P_N}(\rho - 2M\gamma) \leq n_{F_{\mathbb{C}}}(\rho) \leq n_{P_N}(\rho + 2M\gamma),$$

and

$$|n_{F_C}(\varphi; \rho) - n_{F_N}(\varphi; \rho)| \leq CM \cdot \omega\left(\varphi; \frac{2M\gamma}{\rho}\right).$$

□

Remark 3. When applying the previous lemma, the parameters A, B will be arbitrary, but fixed (not depending on ρ). In addition, $\lambda = \log \rho$, and $\gamma = \rho^{-C}$, with some constant $C \geq 4$.

3.4. Logarithmic potential and linear statistics. The following result is known as Jensen's formula.

Theorem 5. *Let $\nu \in \mathcal{M}_1(\mathbb{C})$ and $r > 0$. Then,*

$$U_\nu(0) + \int_0^r \frac{\nu\left(\overline{D(0, t)}\right)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} U_\nu(re^{i\theta}) d\theta.$$

In case ν is a radial measure, we have

$$\int_0^r \frac{\nu\left(\overline{D(0, t)}\right)}{t} dt = U_\nu(r) - U_\nu(0).$$

Proof. This follows from the Poisson-Jensen formula in the disk $D(0, r)$ ([SaT, Theorem 4.10]), applied to the subharmonic function $U_\nu(z)$, and integration by parts. □

For a function $\phi : \mathbb{C} \mapsto \mathbb{R}$, with $\phi_x, \phi_y \in L^2(\mathbb{C})$, we recall

$$\mathfrak{D}(\phi) = \int_{\mathbb{C}} (\phi_x^2 + \phi_y^2) dm(z) = \int_{\mathbb{C}} |\nabla \phi(z)|^2 dm(z).$$

Let $\nu, \mu \in \mathcal{M}_1(\mathbb{C})$ be probability measures with compact support and finite logarithmic energy, and let $\sigma = \nu - \mu$ be a signed measure (with $\sigma(\mathbb{C}) = 0$). It is known ([La, Theorem 1.20], see also [SaT, Proof of Lemma I.1.8]) that

$$\mathfrak{D}(U_\sigma(w)) = \int_{\mathbb{C}} |\nabla U_\sigma(w)|^2 dm(w) = -2\pi \cdot \Sigma(\nu - \mu) < \infty,$$

and in particular that $|\nabla U_\sigma(w)| \in L^2(\mathbb{C})$. We also mention that (as to be expected) $\Sigma(\nu - \mu) \leq 0$, with equality if and only if $\nu = \mu$ ([La, Theorem 1.16], [SaT, Lemma I.1.8]).

The following result allows us to get a lower bound for the ‘distance’ between two measures, in terms of linear statistics (cf. [Pr, (3.10)]).

Lemma 9. *Suppose $\varphi \in C_0^2(\mathbb{C})$ is a compactly supported test function, which is twice continuously differentiable. Let $\nu, \mu \in \mathcal{M}_1(\mathbb{C})$ be probability measures with compact support and finite logarithmic energy. Then*

$$\left| \int_{\mathbb{C}} \varphi(w) d\nu(w) - \int_{\mathbb{C}} \varphi(w) d\mu(w) \right| \leq \frac{1}{\sqrt{2\pi}} \sqrt{\mathfrak{D}(\varphi)} \sqrt{-\Sigma(\nu - \mu)}.$$

Proof. Let us write $\sigma = \nu - \mu$, and recall that $d\sigma(z) = \frac{1}{2\pi} \Delta U_\sigma(z) dm(z)$ in the sense of distributions. Integrating by parts, and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left| \int_{\mathbb{C}} \varphi(w) d\sigma(w) \right| &= \frac{1}{2\pi} \left| \int_{\mathbb{C}} \Delta \varphi(w) U_\sigma(w) dm(w) \right| = \frac{1}{2\pi} \left| \int_{\mathbb{C}} \nabla \varphi(w) \cdot \nabla U_\sigma(w) dm(w) \right| \\ &\leq \frac{1}{2\pi} \sqrt{\int_{\mathbb{C}} |\nabla \varphi(w)|^2 dm(w)} \sqrt{\int_{\mathbb{C}} |\nabla U_\sigma(w)|^2 dm(w)} \\ &= \frac{1}{2\pi} \sqrt{\mathfrak{D}(\varphi)} \sqrt{\mathfrak{D}(U_\sigma(w))} = \frac{1}{\sqrt{2\pi}} \sqrt{\mathfrak{D}(\varphi)} \sqrt{-\Sigma(\nu - \mu)}. \end{aligned}$$

□

4. PROBABILITY OF LARGE FLUCTUATIONS IN THE NUMBER OF ZEROS - UPPER BOUND

Given $p \geq 0$, $p \neq 1$, we find in this section an asymptotic upper bound for the probability of the event $\mathbb{P}[n(r) = \lfloor pr^2 \rfloor]$, as $r \rightarrow \infty$, where $n(r) = n_{F_{\mathbb{C}}}(r)$ is the number of zeros of the GEF inside the disk $\{|z| \leq r\}$. Recall the GEF is given by the random Taylor series

$$F_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}, \quad z \in \mathbb{C},$$

where $\{\xi_k\}$ is a sequence of independent standard complex Gaussians. Almost surely all of the zeros of $F_{\mathbb{C}}$ are simple, so we can ignore multiplicities in this paper. The well-known Edelman-Kostlan formula ([HoKPV, Section 2.4]) implies that the mean number of zeros of per unit area is $\frac{1}{\pi}$, and in particular $\mathbb{E}[n(r)] = r^2$. The asymptotic behavior of the variance $\text{Var}[n(r)]$ was originally computed by Forrester and Honner in [FoH]:

$$\text{Var}[n(r)] = \kappa_1 r + o(r), \quad r \rightarrow \infty,$$

with an explicit constant κ_1 . It was shown in the paper [NaS2] that the normalized random variables $\frac{n(r) - r^2}{\sqrt{\text{Var}[n(r)]}}$ converge in distribution to a standard Gaussian random variable (asymptotic normality).

We wish to find the precise logarithmic asymptotics for the probability $n(r)$ is (very) far from its expected value. To formulate our theorem, we define a function $q(p) : [0, \infty) \rightarrow [0, e]$ as follows:

- (1) In case $p \in (0, 1) \cup (1, e)$, take $q \neq p$ to be the solution of $p(\log p - 1) = q(\log q - 1)$.
- (2) In the remaining cases, put $q = e$ for $p = 0$, $q = 1$, for $p = 1$, and $q = 0$ for $p \geq e$.

Note that q can be written explicitly in terms of p , using the Lambert function [CoGHJK].

Theorem 6. Fix $p \in [0, \infty) \setminus \{1\}$. With $q = q(p)$ as above, we have as $r \rightarrow \infty$

$$\mathbb{P}[n(r) = \lfloor pr^2 \rfloor] = \exp(-Z_p r^4 + O(r^2 \log^2 r)),$$

where

$$Z_p = \left| \int_p^{q(p)} x \log x dx \right|.$$

Remark 4. A simple calculation shows

$$H_p = \begin{cases} \left| \frac{1}{4} [q^2 (2 \log q - 1) - p^2 (2 \log p - 1)] \right| & p \in (0, e) \setminus \{1\}; \\ \frac{1}{4} p^2 (2 \log p - 1) & p \geq e. \end{cases}$$

The case $H_0 = \frac{e^2}{4}$, corresponding to the hole event $\{n(r) = 0\}$, follows from the results in [Ni3] (with a slightly better error term). See Section 8 for a graph of this function.

In this section we obtain the upper bound in Theorem 6. The proof is slightly more general than necessary for proving the results of this section, as we are going to use it again in Section 7. In Section 6, we prove the lower bound in Theorem 6.

Remark 5. Many parameters appear in the course of the proof. Ultimately they all depend on p and r , that appear in the statement of Theorem 6. The parameter $B \geq 1$ is fixed (but can be arbitrarily large), the parameter A depends only on p , for the other parameters we have

$$\gamma = t = r^{-C_2}, \lambda = \log r, L = r + O\left(\frac{1}{r}\right),$$

where $C_2 \geq 4$ is fixed. In addition,

$$N \in \{\lfloor \lambda r^2 \rfloor + 1, \lfloor 2\lambda r^2 \rfloor + 1\}, \alpha = Nr^{-2} \leq 3 \log r.$$

4.1. Truncation of the power series. Let $B \geq 1$ be a fixed constant, and suppose that $r > 0$ is large. Denote by $n(r) = n_{F_C}(r)$ the number of zeros of the GEF inside the disk $D(0, r)$. We wish to approximate F_C by a polynomial, in such a way that the zeros of the polynomial are close to the zeros of the GEF inside the larger disk $D(0, Br)$.

Suppose φ is a continuous test function supported on the disk $D(0, B)$, where $B \geq 1$. Let $A \geq 1$, $\lambda > 16$, and $\gamma = r^{-C_2}$, with $C_2 \geq 4$. In addition, put $N_0 = \lfloor \lambda r^2 \rfloor + 1$, $N_1 = \lfloor 2\lambda r^2 \rfloor + 1$. We now wish to apply Lemma 7 and Lemma 8 with $\rho = r$. We notice that if $A = O(1)$, $\lambda = \log r$, then the conditions of both lemmas are satisfied. We find that there exist events E_{reg} and E_{reg}^N , $N \in \{N_0, \dots, N_1\}$, such that

$$E_{\text{reg}} = \bigcup_{N=N_0}^{N_1} E_{\text{reg}}^N, \quad \mathbb{P}[E_{\text{reg}}^c] \leq \exp(-C \cdot AB^4 r^4).$$

Put $M_0 = 8B^3 r^3$, and $K_0 = 2M_0 \gamma \leq Cr^{3-C_2} = O\left(\frac{1}{r}\right)$. If we write

$$P_N(z) = \sum_{k=0}^N \xi_k \frac{z^k}{\sqrt{k!}}, \quad z \in \mathbb{C},$$

then, on the event E_{reg}^N , we have

$$(4.1) \quad n_{P_N}(r - K_0) \leq n(r) \leq n_{P_N}(r + K_0), \quad K_0 = 16B^3 r^3 \gamma \leq Cr^{3-C_2} = O\left(\frac{1}{r}\right),$$

$$(4.2) \quad |n_{F_C}(\varphi; r) - n_{P_N}(\varphi; r)| \leq CM_0 \cdot \omega(\varphi; K_0 r^{-1}),$$

$$\text{and } \frac{1}{|\xi_N|} \sum_{k=0}^N |\xi_k|^2 \leq \exp(Cr^2).$$

4.1.1. Choice of the parameter A . Let $n(r)$ be the number of zeros of the GEF $F_C(z)$ in the disk $\{|z| \leq r\}$. We assume that one of the following two events occurs:

Case 1. $n(r) \leq pr^2$, where $p \in [0, 1)$.

Case 2. I: $n(r) \geq pr^2$, where $p \in (1, e)$. II: $n(r) \geq pr^2$, where $p \in [e, \infty)$.

We always assume that r is sufficiently large for all the different asymptotic estimates that we use (this might depend on p in Case 2.II.). We remark that, if $C > 0$ is a sufficiently large numerical constant, then events with probability at most $\exp(-Cr^4)$ would be negligible events in Case 1 and Case 2.I.

In Case 2.II. the same holds with $\exp(-Cp^2 \log p \cdot r^4)$. Let $C_1 > 0$ be a sufficiently large numerical constant, we then set

$$A = C_1 (1 \vee p^2 \log p).$$

We choose C_1 such that the event E_{reg}^c is negligible.

4.1.2. Scaling of the polynomial. *The parameters L and α .* Let $L > 0$ be a large parameter. For $N \in \{N_0, \dots, N_1\}$ we set $\alpha = Nr^{-2}$, and remark that $\alpha \leq 3 \log r$. We will pick the precise value of L later, but in all cases it holds that $L = r + O(\frac{1}{r})$, and therefore $L^2 = r^2 + O(1)$. In the rest of the section, it will be more convenient to consider the scaled polynomials

$$P_{N,L}(z) = \sum_{k=0}^N \xi_k \frac{(Lz)^k}{\sqrt{k!}}.$$

Rewriting (4.1) and (4.2) in terms of $P_{N,L}$, we get

$$(4.3) \quad n_{P_{N,L}}\left(\frac{r - K_0}{L}\right) \leq n(r) \leq n_{P_{N,L}}\left(\frac{r + K_0}{L}\right),$$

$$(4.4) \quad \left| n_{F_C}(\varphi; r) - n_{P_{N,L}}\left(\varphi; \frac{r}{L}\right) \right| \leq CM_0 \cdot \omega(\varphi; K_0 r^{-1}).$$

4.2. Estimates for the joint distribution of the zeros of $P_{N,L}$. We denote the zeros of the polynomial $P_{N,L}$ by z_1, \dots, z_N (in uniform random order). In many cases it will be convenient to use the vector notation $\underline{z} = (z_1, \dots, z_N)$. Recall that $N = \alpha L^2 + O(\alpha)$. We will frequently use the notation $|\Delta(\underline{z})|^2 = \prod_{j \neq k} |z_j - z_k|$ and the (probability) measure

$$d\mu_L(w) = \frac{L^2}{\pi} e^{-L^2|w|^2} dm(w),$$

where m is Lebesgue measure on \mathbb{C} . Using a change of variables from the coefficients of the polynomial $P_{N,L}$ to the zeros (see Lemma 11 in Appendix A), we find that the joint distribution of the zeros, w.r.t. $m(\underline{z})$, the Lebesgue measure on \mathbb{C}^N , is given by

$$(4.5) \quad f(\underline{z}) = f(z_1, \dots, z_N) = A_L^N |\Delta(\underline{z})|^2 \left(\int_{\mathbb{C}} \prod_{j=1}^N |w - z_j|^2 d\mu_L(w) \right)^{-(N+1)},$$

where the normalizing constant A_L^N is given by

$$(4.6) \quad \begin{aligned} A_L^N &= \frac{N! \cdot \prod_{j=1}^N j!}{\pi^N L^{N(N+1)}} = \exp\left(\frac{1}{2}N^2 \log\left(\frac{N}{L^2}\right) - \frac{3}{4}N^2 + O(N(\log N + \log L))\right) \\ &= \exp\left(\frac{1}{2}N^2 \log\left(\frac{N}{L^2}\right) - \frac{3}{4}N^2 + O(L^2 \log^2 L)\right), \end{aligned}$$

where we used $\alpha \leq 3 \log r = 3 \log L + O(1)$. We denote by $q_{\underline{z}}(w) = \prod_{j=1}^N (w - z_j)$ the monic polynomial corresponding to $P_{N,L}$. By Lemma 12, we have

$$S(\underline{z}) \stackrel{\text{def}}{=} \int_{\mathbb{C}} |q_{\underline{z}}(w)|^2 d\mu_L(w) \geq \sup_{w \in \mathbb{C}} \left\{ |q_{\underline{z}}(w)|^2 e^{-L^2|w|^2} \right\} \stackrel{\text{def}}{=} A(\underline{z}).$$

4.2.1. *Estimates for $A(\underline{z})$ and $S(\underline{z})$.* In order to control the convergence of the integral over the density (4.5), we need a simple lower bound for $A(\underline{z})$. We will use the identity ([SaT, Example 0.5.7])

$$(4.7) \quad \frac{1}{2\pi} \int_0^{2\pi} \log |te^{i\theta} - z| \, d\theta = \log t \vee |z|.$$

Claim 1. We have

$$A(\underline{z}) \geq \left[\prod_{j=1}^N 1 \vee |z_j| \right]^2 \exp(-L^2).$$

Proof. Using (4.7) with $t = 1$, we find

$$\begin{aligned} \frac{1}{2} \log A(\underline{z}) &= \sup_{w \in \mathbb{C}} \left\{ \log |q_{\underline{z}}(w)| - \frac{L^2}{2} |w|^2 \right\} \geq \frac{1}{2\pi} \int_0^{2\pi} \log |q_{\underline{z}}(e^{i\theta})| \, d\theta - \frac{L^2}{2} \\ &= \sum_{j=1}^N \log(1 \vee |z_j|) - \frac{L^2}{2}. \end{aligned}$$

□

Now we clearly have,

Claim 2. For $b > 1$,

$$\int_{\mathbb{C}^N} A(\underline{z})^{-b} \, dm(\underline{z}) \leq \exp(bL^2) \cdot \left(\frac{Cb}{b-1} \right)^N.$$

Proof. By the previous claim,

$$\begin{aligned} \int_{\mathbb{C}^N} A(\underline{z})^{-b} \, dm(\underline{z}) &\leq \exp(bL^2) \cdot \int_{\mathbb{C}^N} \left[\prod_{j=1}^N 1 \vee |z_j| \right]^{-2b} \, dm(\underline{z}) \\ &= \exp(bL^2) \cdot \left[\int_{\mathbb{C}} (1 \vee |z|)^{-2b} \, dm(z) \right]^N \\ &\leq \exp(bL^2) \cdot \left(\frac{Cb}{b-1} \right)^N. \end{aligned}$$

□

We will also need a probabilistic upper bound for $S(\underline{z})$.

Claim 3. On the event E_{reg}^N we have

$$S(\underline{z}) \leq \exp(C\alpha \log \alpha \cdot L^2).$$

Proof. In Claim 13, Appendix A we show

$$\int_{\mathbb{C}} |q_{\underline{z}}(w)|^2 \, d\mu_L(w) = \left(\sum_{k=0}^N |\xi_k|^2 \right) \cdot \left(|\xi_N|^2 \frac{L^{2N}}{N!} \right)^{-1}.$$

On the event E_{reg}^N (and using $N! \leq N^N$) we have,

$$\left(|\xi_N|^{-2} \sum_{k=0}^N |\xi_k|^2 \right) \cdot \left(\frac{L^{2N}}{N!} \right)^{-1} \leq \exp(Cr^2) \left(\frac{N}{L^2} \right)^N \leq \exp(CL^2 + C\alpha L^2 \log \alpha) \leq \exp(C\alpha \log \alpha \cdot L^2).$$

□

4.2.2. *Upper bound for the probability.* Consider a set $Z \subset \mathbb{C}^N$. We think about Z as a collection of possible ‘configurations’ of the zeros of $P_{N,L}$. We are interested in bounding the probability of these configurations. We introduce the functional $I^* : \mathbb{C}^N \rightarrow \mathbb{R}$:

$$I^*(\underline{z}) = \begin{cases} 2 \sup_{w \in \mathbb{C}} \left\{ \frac{1}{N} \cdot \log |q_{\underline{z}}(w)| - \frac{L^2}{2N} |w|^2 \right\} - \frac{1}{N^2} \sum_{j \neq k} \log |z_j - z_k| & \forall j \neq k, z_j \neq z_k; \\ \infty & \text{otherwise.} \end{cases}$$

We will show, that at the exponential scale, the probability of the configurations we consider is bounded above by the minimum of the functional I^* over Z .

Starting with the joint density of the zeros (4.5), we find that

$$\mathbb{P}[Z \cap E_{\text{reg}}^N] \leq A_L^N \int_E |\Delta(\underline{z})|^2 S(\underline{z})^{-(N+1)} dm(\underline{z}),$$

where

$$E = Z \cap \{\underline{z} \in \mathbb{C}^N : S(\underline{z}) \leq \exp(C\alpha \log \alpha \cdot L^2)\}.$$

Notice that on E_{reg}^N

$$\begin{aligned} S(\underline{z})^{-(N+1)} &\leq A(\underline{z})^{-(1+\frac{1}{N})} S(\underline{z})^{\frac{1}{N}} A(\underline{z})^{-N} \\ &\leq A(\underline{z})^{-(1+\frac{1}{N})} \exp(C \log \alpha) \cdot \exp(-N \cdot \log A(\underline{z})). \end{aligned}$$

Therefore, by Claim 2 (with $b = 1 + \frac{1}{N}$) and (4.6) we have

$$\begin{aligned} \mathbb{P}[Z \cap E_{\text{reg}}^N] &\leq A_L^N \cdot \exp(C \log \alpha) \cdot \int_E \exp\left(\sum_{j \neq k} \log |z_j - z_k| - N \cdot \log A(\underline{z})\right) A(\underline{z})^{-(1+\frac{1}{N})} dm(\underline{z}) \\ &\leq A_L^N \cdot \exp(C \log \alpha) \cdot (CN)^N \cdot \exp\left(-N^2 \cdot \inf_{\underline{z} \in E} I^*(\underline{z})\right) \\ &\leq \exp\left(-N^2 \cdot \inf_{\underline{z} \in Z} I^*(\underline{z}) + \frac{1}{2} N^2 \log\left(\frac{N}{L^2}\right) - \frac{3}{4} N^2 + O(L^2 \log^2 L)\right). \end{aligned}$$

Let $\nu \in \mathcal{M}_1(\mathbb{C})$ be a probability measure. We now introduce the functional

$$I_\alpha(\nu) = 2 \sup_{w \in \mathbb{C}} \left\{ U_\nu(w) - \frac{|w|^2}{2\alpha} \right\} - \Sigma(\nu),$$

where $U_\nu(w)$ and $\Sigma(\nu)$ are the logarithmic potential and the logarithmic energy of the measure ν , respectively (see notation in Section 1). We discuss this functional in more details in Section 5.

Define the empirical probability measure of the zeros by

$$\mu_{\underline{z}} = \frac{1}{N} \sum_{j=1}^N \delta_{z_j},$$

where δ_z is a Dirac delta measure at the point $z \in \mathbb{C}$. A technical issue is the fact that the logarithmic energy of the empirical measure is not defined. To resolve it, we smoothen the empirical measure by defining

$$\mu_{\underline{z}}^t = \mu_{\underline{z}} \star m_{|z|=t} = \frac{1}{N} \sum_{j=1}^N m_{|z-z_j|=t},$$

where $m_{|z-w|=t}$ is the (normalized) Lebesgue measure on the circle $|z-w| = t$. We now wish to compare $I^*(z)$ and $I_\alpha(\mu_z^t)$.

4.2.3. *Comparing $I^*(z)$ and $I_\alpha(\mu_z^t)$.* We will now show that for $t > 0$ sufficiently small we have

$$I^*(z) \geq I_\alpha(\mu_z^t) - C \left(\frac{1}{N} \log \frac{1}{t} + \frac{t}{\sqrt{\alpha}} + \frac{1}{L^2} \right).$$

The proof consists of two simple claims.

Claim 4. We have

$$\frac{1}{N^2} \sum_{j \neq k} \log |z_j - z_k| \leq \Sigma(\mu_z^t) + \frac{C \log \frac{1}{t}}{N}.$$

Proof. Note that

$$\int \log |z - w| \, dm_{|w-a|=t} = \frac{1}{2\pi} \int_0^{2\pi} \log |z - a + te^{i\theta}| \, d\theta = \log |z - a| \vee t.$$

Since the logarithm is a subharmonic function, we have

$$\begin{aligned} \frac{1}{N^2} \sum_{j \neq k} \log |z_j - z_k| &\leq \frac{1}{N^2} \sum_{j \neq k} \int \log |z - w| \, dm_{|z-z_j|=t} dm_{|w-z_k|=t} \\ &\leq \frac{1}{N^2} \sum_{j,k=1}^N \int \log |z - w| \, dm_{|z-z_j|=t} dm_{|w-z_k|=t} + \frac{C \log \frac{1}{t}}{N} \\ &= \int \log |z - w| \, d\mu_z^t(z) d\mu_z^t(w) + \frac{C \log \frac{1}{t}}{N}. \end{aligned}$$

□

Let us write $B_\alpha(\nu) = 2 \cdot \sup \left\{ U_\nu(w) - \frac{|w|^2}{2\alpha} : w \in \mathbb{C} \right\}$. By Claim 14 in Appendix B, we have

$$B_\alpha(\nu) = 2 \sup_{|w| \leq \sqrt{\alpha}} \left\{ U_\nu(w) - \frac{|w|^2}{2\alpha} \right\}.$$

Claim 5. For $t > 0$ sufficiently small, we have

$$\frac{1}{N} \log A(z) = 2 \sup_{w \in \mathbb{C}} \left\{ \frac{1}{N} \log |q_z(w)| - \frac{L^2}{2N} |w|^2 \right\} \geq B_\alpha(\mu_z^t) - C \left(\frac{1}{L^2} + \frac{t}{\sqrt{\alpha}} \right).$$

Proof. We can write

$$B_\alpha(\mu_z^t) = 2 \sup_{|w| \leq \sqrt{\alpha}} \left\{ U_{\mu_z^t}(w) - \frac{|w|^2}{2\alpha} \right\} = 2 \sup_{|w| \leq \sqrt{\alpha}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} U_{\mu_z}(w + te^{i\theta}) \, d\theta - \frac{|w|^2}{2\alpha} \right\}.$$

By the definition of $A(z)$, for any $\theta \in [0, 2\pi]$,

$$\begin{aligned} U_{\mu_z}(w + te^{i\theta}) - \frac{L^2}{2N} |w + te^{i\theta}|^2 &= \frac{1}{N} \sum_{j=1}^N \log |w + te^{i\theta} - z_j| - \frac{L^2}{2N} |w + te^{i\theta}|^2 \\ &= \frac{1}{N} \log |q_z(w + te^{i\theta})| - \frac{L^2}{2N} |w + te^{i\theta}|^2 \leq \frac{1}{2N} \log A(z). \end{aligned}$$

Recall that $\frac{N}{L^2} = \frac{\alpha r^2}{L^2} = \alpha (1 + O(L^{-2}))$. Notice that for $|w| \leq \sqrt{\alpha}$ and for $t \leq 1$, we have

$$\frac{L^2}{2N} |w + te^{i\theta}|^2 \leq \frac{L^2}{2N} |w|^2 + \frac{Ct}{\sqrt{\alpha}} \leq \frac{|w|^2}{2\alpha} \left(\frac{\alpha L^2}{N} \right) + \frac{Ct}{\sqrt{\alpha}} \leq \frac{|w|^2}{2\alpha} + \frac{C}{L^2} + \frac{Ct}{\sqrt{\alpha}}.$$

Hence,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} U_{\mu_{\underline{z}}}(w + te^{i\theta}) d\theta - \frac{|w|^2}{2\alpha} &\leq \frac{1}{2\pi} \int_0^{2\pi} \left[U_{\mu_{\underline{z}}}(w + te^{i\theta}) - \frac{L^2}{2N} |w + te^{i\theta}|^2 \right] d\theta + \frac{C}{L^2} + \frac{Ct}{\sqrt{\alpha}} \\ &\leq \frac{1}{N} \sup_{w \in \mathbb{C}} \left\{ \log |q_{\underline{z}}(w)| - \frac{1}{2} L^2 |w|^2 \right\} + \frac{C}{L^2} + \frac{Ct}{\sqrt{\alpha}}. \end{aligned}$$

□

4.3. Reduction to a modified weighted energy problem. For a set $Z_N \subset \mathbb{C}^N$, after combining the estimates above (and using $N = \alpha L^2 + O(\alpha)$, $\alpha \leq C \log L$), we find that

(4.8)

$$\mathbb{P}[Z_N \cap E_{\text{reg}}^N] \leq \exp \left(-N^2 \left[\inf_{\underline{z} \in Z_N} I_{\alpha}(\mu_{\underline{z}}^t) - \frac{1}{2} \log \left(\frac{N}{L^2} \right) + \frac{3}{4} \right] + L^2 \log L \cdot O \left(\log L + \log \frac{1}{t} + t L^2 \sqrt{\log L} \right) \right).$$

4.3.1. Choosing the parameters t and L . We now choose $t = r^{-C_2}$ (with $C_2 \geq 4$) and recall that $\gamma = r^{-C_2}$, $K_0 = 16B^3 r^3 \gamma \leq C r^{3-C_2} = O(\frac{1}{r})$. Consider again the two cases we described at the beginning of the section. In Case 1, we set $L = (1+t)^{-1}(r - K_0)$. Since $L = r + O(\frac{1}{r})$, and using (4.3) we find

$$(4.9) \quad n_{P_{N,L}}(1+t) = n_{P_N}(r - K_0) \leq n(r) \leq pr^2 = \frac{pN}{\alpha} \implies \mu_{\underline{z}}^t(D) \leq \frac{p}{\alpha}.$$

Similarly, in Case 2, we set $L = (1-t)^{-1}(r + K_0)$, and get

$$(4.10) \quad n_{P_{N,L}}(1-t) = n_{P_N}(r + K_0) \geq n(r) \geq pr^2 = \frac{pN}{\alpha} \implies \mu_{\underline{z}}^t(\overline{D}) \geq \frac{p}{\alpha}.$$

Define the set

$$L_{\varphi, \tau, \lambda}^N = L_{\varphi, \tau, \lambda}^N(t) = \left\{ \underline{z} : \left| \frac{1}{N} \sum_{j=1}^N \varphi(z_j) - \tau \right| \geq \lambda + \omega(\varphi; t) \right\}.$$

Notice that $|\varphi(z_j) - \int \varphi d\mu_{|z-z_j|=t}| \leq \omega(\varphi; t)$, hence, if $\underline{z} \in L_{\varphi, \tau, \lambda}^N$, then

$$\left| \int_{\mathbb{C}} \varphi(w) d\mu_{\underline{z}}^t(w) - \tau \right| \geq \lambda.$$

4.3.2. Completing the reduction. For $x \geq 0$, we define the following sets of measures:

$$\begin{aligned} \mathcal{F}_x &= \left\{ \nu \in \mathcal{M}_1(\mathbb{C}) : \nu(D) \leq \frac{x}{\alpha} \right\}, \\ \mathcal{M}_x &= \left\{ \nu \in \mathcal{M}_1(\mathbb{C}) : \nu(\overline{D}) \geq \frac{x}{\alpha} \right\}, \\ \mathcal{L}_{\varphi, \tau, x} &= \left\{ \nu \in \mathcal{M}_1(\mathbb{C}) : \left| \int_{\mathbb{C}} \varphi(w) d\nu(w) - \tau \right| \geq x \right\}. \end{aligned}$$

Clearly we have,

$$(4.11) \quad \left\{ \mu_{\underline{z}}^t : \mu_{\underline{z}}^t(D) \leq \frac{p}{\alpha} \right\} \subset \mathcal{F}_p, \left\{ \mu_{\underline{z}}^t : \mu_{\underline{z}}^t(\overline{D}) \geq \frac{p}{\alpha} \right\} \subset \mathcal{M}_p, \left\{ \mu_{\underline{z}}^t : \underline{z} \in L_{\varphi, \tau, \lambda}^N \right\} \subset \mathcal{L}_{\varphi, \tau, \lambda}.$$

We thus reduced an estimate for the probability, to a minimization problem for a functional acting on (general) probability measures.

4.4. The upper bound in Theorem 6. We remind the two cases of the theorem:

Case 1. $n(r) \leq pr^2$, where $p \in [0, 1)$.

Case 2. I: $n(r) \geq pr^2$, where $p \in (1, e)$. II: $n(r) \geq pr^2$, where $p \in [e, \infty)$.

Recall $\alpha \leq 3 \log r$, $t = r^{-C_2}$ ($C_2 \geq 4$), $L = r + O(\frac{1}{r})$, and $N = \alpha r^2 = \alpha L^2 + O(\alpha)$. In Case 1, using (4.8), (4.9) and the first inclusion (4.11), we get

$$\begin{aligned} \log \mathbb{P} \left[\{n(r) \leq pr^2\} \cap E_{\text{reg}}^N \right] &\leq \log \mathbb{P} \left[\left\{ n_{P_{N,L}}(1+t) \leq \frac{pN}{\alpha} \right\} \cap E_{\text{reg}}^N \right] \\ &\leq -N^2 \left[\inf_{\nu \in \mathcal{F}_p} I_{\alpha}(\nu) - \frac{1}{2} \log \left(\frac{N}{L^2} \right) + \frac{3}{4} \right] + L^2 \log L \cdot O \left(\log L + \log \frac{1}{t} + tL^2 \sqrt{\log L} \right) \\ &= -N^2 \left[\inf_{\nu \in \mathcal{F}_p} I_{\alpha}(\nu) - \frac{1}{2} \log \alpha + \frac{3}{4} \right] + O \left(\frac{N^2}{L^2} + r^2 \log^2 r \right) \\ &= -N^2 \left[\inf_{\nu \in \mathcal{F}_p} I_{\alpha}(\nu) - \frac{1}{2} \log \alpha + \frac{3}{4} \right] + O \left(r^2 \log^2 r \right). \end{aligned}$$

In (5.1), Subsection 5.2.1, we show that the minimal value of the functional $I_{\alpha}(\nu)$ over the set $\mathcal{F}_p = \{\nu \in \mathcal{M}_1(\mathbb{C}) : \nu(D) \leq \frac{p}{\alpha}\}$ is given by

$$\inf_{\nu \in \mathcal{F}_p} I_{\alpha}(\nu) = \frac{1}{2} \log \alpha - \frac{3}{4} + \frac{q^2(2 \log q - 1)}{4\alpha^2} - \frac{p^2(2 \log p - 1)}{4\alpha^2},$$

where $q = q(p) > 1$ is the solution of the equation $q(\log q - 1) = p(\log p - 1)$.

Summing up over $N \in \{N_0, \dots, N_1\}$, we get

$$\begin{aligned} \mathbb{P} \left[\{n(r) \leq pr^2\} \cap E_{\text{reg}} \right] &\leq \sum_{N=N_0}^{N_1} \mathbb{P} \left[\left\{ n_{P_{N,L}}(1+t) \leq \frac{pN}{\alpha} \right\} \cap E_{\text{reg}}^N \right] \\ &\leq \sum_{N=N_0}^{N_1} \exp \left(-\frac{N^2}{4\alpha^2} [q^2(2 \log q - 1) - p^2(2 \log p - 1)] + O(r^2 \log^2 r) \right) \\ &= \exp \left(-\frac{r^4}{4} [q^2(2 \log q - 1) - p^2(2 \log p - 1)] + O(r^2 \log^2 r + \log r) \right) \\ (4.12) \quad &\leq \exp \left(-\frac{1}{4} [q^2(2 \log q - 1) - p^2(2 \log p - 1)] r^4 + O(r^2 \log^2 r) \right). \end{aligned}$$

Notice that E_{reg}^c is a negligible event, and therefore we can use the simple bound $\mathbb{P}[\{n(r) \leq pr^2\}] \leq \mathbb{P}[\{n(r) \leq pr^2\} \cap E_{\text{reg}}] + \mathbb{P}[E_{\text{reg}}^c]$. The upper bounds in Case 2.I. and Case 2.II. are obtained in a similar way. We leave the details for the reader. This completes the proof of the upper bound in Theorem 6.

5. MODIFIED ENERGY PROBLEMS

Let $\nu \in \mathcal{M}_1(\mathbb{C})$ be a probability measure and $\alpha > 0$. In this section we consider the problem of minimizing the functional

$$I_\alpha(\nu) = 2 \sup_{w \in \mathbb{C}} \left\{ U_\nu(w) - \frac{|w|^2}{2\alpha} \right\} - \Sigma(\nu) \stackrel{\text{def}}{=} B_\alpha(\nu) - \Sigma(\nu),$$

over probability measures with compact support, restricted to certain (closed and convex) subsets of $\mathcal{M}_1(\mathbb{C})$. We mention that this functional is lower semi-continuous and strictly convex. It is known that $\Sigma(\nu)$ is an upper semi-continuous and strictly concave functional ([HiP, Proposition 2.2]). Note that $B_\alpha(\nu)$ is lower semi-continuous and convex as the supremum of affine functionals. This implies that a unique minimizer of $I_\alpha(\nu)$ exists over closed and convex subsets of $\mathcal{M}_1(\mathbb{C})$. In a more general setting, it is proved in [ZeZ, Lemma 29] that the global minimizer of $I_\alpha(\nu)$ is the uniform probability measure on the disk $D(0, \sqrt{\alpha})$.

It will be useful to make the following definition

$$g_\nu(z) = U_\nu(z) - \frac{|z|^2}{2\alpha} - \frac{B_\alpha(\nu)}{2}.$$

Notice that $g_\nu(z) \leq 0$ for all $z \in \mathbb{C}$.

Remark 6. Since in our case the minimizers turn out to be compactly supported, it is enough to identify the minimizing measure among measures with compact support (since we can approximate an arbitrary measure using measures with compact support). In general, some energy problems can have solutions which are not compactly supported, even in case the global minimizer has compact support (e.g., [GhZ, Theorem 1.3]).

5.1. The principle of domination. The key tool that we need from potential theory is a special case of the principle of domination ([SaT, pg. 43]).

Theorem 7. *Let ν and μ be probability measures with compact support and finite logarithmic energy. If*

$$U_\nu(z) \leq U_\mu(z) + C, \quad \mu - \text{a.e. in } z,$$

then

$$U_\nu(z) \leq U_\mu(z) + C, \quad \forall z \in \mathbb{C}.$$

We apply it using the next claim (cf. [ZeZ, Lemma 29]).

Claim 6. Let ν and μ be probability measures with compact support and finite logarithmic energy. If $\mu - \text{a.e. in } z$ we have $g_\nu(z) \leq g_\mu(z)$, then $I_\alpha(\mu) \leq I_\alpha(\nu)$.

Proof. By the assumption and using the previous theorem, we find

$$U_\nu(z) - \frac{B_\alpha(\nu)}{2} \leq U_\mu(z) - \frac{B_\alpha(\mu)}{2}, \quad z \in \mathbb{C}.$$

Now,

$$\begin{aligned} \Sigma(\nu) &= \int_{\mathbb{C}} U_\nu(z) \, d\nu(z) \leq \int_{\mathbb{C}} U_\mu(z) \, d\nu(z) + \frac{B_\alpha(\nu) - B_\alpha(\mu)}{2} = \int_{\mathbb{C}} U_\nu(z) \, d\mu(z) + \frac{B_\alpha(\nu) - B_\alpha(\mu)}{2} \\ &\leq \int_{\mathbb{C}} U_\mu(z) \, d\mu(z) + B_\alpha(\nu) - B_\alpha(\mu) = \Sigma(\mu) + B_\alpha(\nu) - B_\alpha(\mu). \end{aligned}$$

□

Therefore, in order to establish that a certain measure μ minimizes the value of the functional $I_\alpha(\nu)$ (maybe over some subset of $\mathcal{M}_1(\mathbb{C})$), it is sufficient to show that for any other measure ν , the inequality $g_\nu(z) \leq g_\mu(z)$ is satisfied on the support of μ (this is mainly useful for problems where the minimizer is a radially symmetric measure).

5.2. Identifying the minimizing measures. We consider here the solution of rotation symmetric minimization problems for the functional $I_\alpha(\nu)$ (that is, a rotation of a measure that satisfies the constraint continues to satisfy it). If μ is a measure that satisfies a rotation symmetric constraint, then its radial symmetrization μ_{rad} also satisfies this constraint (we define μ_{rad} as the normalized integral over all the rotations of μ w.r.t. the origin). By the radial symmetry and the convexity of the functional $I_\alpha(\nu)$, we have that $I_\alpha(\mu_{\text{rad}}) \leq I_\alpha(\mu)$. This implies the solution is a radial measure, and we will consider these minimization problems over the set of radially symmetric measures.

5.2.1. *The case $p \in [0, 1)$.* We now consider the minimization problem over the set of measures

$$\mathcal{F}_p = \mathcal{F}_{p,\alpha} = \left\{ \nu \in \mathcal{M}_1(\mathbb{C}) : \nu(D) \leq \frac{p}{\alpha} \right\}.$$

Notice that this is a closed set of measures, since D is an open set. It is also clearly convex.

We define $q = q(p) > 1$ as the solution of the equation $q(\log q - 1) = p(\log p - 1)$. We now show that the minimizing measure is given by

$$\mu_{Z_p}(z) = \mu_{Z_p}^\alpha(z) = \frac{1}{\alpha} \left[\mathbf{1}_{\{|z| \leq \sqrt{p}\}}(z) + \mathbf{1}_{\{\sqrt{q} \leq |z| \leq \sqrt{\alpha}\}}(z) \right] \cdot \frac{m(z)}{\pi} + \frac{q-p}{\alpha} m_{\{|z|=1\}},$$

where $m_{\{|z|=1\}}$ is the normalized Lebesgue measure on $\{|z|=1\}$, and m is Lebesgue measure on \mathbb{C} . After a straightforward computation we find the logarithmic potential of μ_{Z_p} (on its support) to be:

$$U_{\mu_{Z_p}}(z) = \frac{\log \alpha}{2} - \frac{1}{2} + \begin{cases} \frac{|z|^2}{2\alpha} & 0 \leq |z| \leq \sqrt{p}; \\ \frac{q(1-\log q)}{2\alpha} & |z| = 1; \\ \frac{|z|^2}{2\alpha} & \sqrt{q} \leq |z| \leq \sqrt{\alpha}. \end{cases}$$

In addition, it is easy to verify the following results

$$(5.1) \quad B(\mu_{Z_p}) = \log \alpha - 1, \quad I_\alpha(\mu_{Z_p}) = \frac{\log \alpha}{2} - \frac{3}{4} + \frac{q^2(2\log q - 1)}{4\alpha^2} - \frac{p^2(2\log p - 1)}{4\alpha^2}.$$

These computations imply $g_{\mu_{Z_p}}(z) = 0$ on the support of μ_{Z_p} except for $|z| = 1$, and there we have $g_{\mu_{Z_p}}(z) = \frac{q(1-\log q)}{2\alpha} - \frac{1}{2\alpha}$. It remains to prove the following simple result.

Claim 7. Let ν be a radially symmetric measure with compact support. If $\nu(D) \leq \frac{p}{\alpha}$, then $g_\nu(z) \leq \frac{q(1-\log q)}{2\alpha} - \frac{1}{2\alpha}$ for all z with $|z| = 1$.

Proof. Since ν is radial we have by Jensen's formula (5),

$$U_\nu(1) - U_\nu(\sqrt{p}) = \int_{\sqrt{p}}^1 \frac{\nu(|z| \leq t)}{t} dt \leq \frac{p}{\alpha} \int_{\sqrt{p}}^1 \frac{1}{t} dt = -\frac{p \log p}{2\alpha}.$$

Now,

$$\begin{aligned} g_\nu(z) &= g_\nu(1) = U_\nu(1) - \frac{1}{2\alpha} - \frac{B_\alpha(\nu)}{2} \leq U_\nu(1) - \frac{1}{2\alpha} - \left[U_\nu(\sqrt{p}) - \frac{p}{2\alpha} \right] \\ &\leq \frac{p(1-\log p)}{2\alpha} - \frac{1}{2\alpha} = \frac{q(1-\log q)}{2\alpha} - \frac{1}{2\alpha}. \end{aligned}$$

□

Since for any radially symmetric measure ν with compact support we have $g_\nu(z) \leq g_{\mu_{Z_p}}(z)$ on the support of μ_{Z_p} , Claim 6 implies that μ_{Z_p} is the minimizing measure over the set \mathcal{F}_p .

5.2.2. *The case $p > 1$.* In a similar way to the previous problem, we now consider the minimization problem over the convex set of measures

$$\mathcal{M}_p = \mathcal{M}_{p,\alpha} = \left\{ \nu \in \mathcal{M}_1(\mathbb{C}) : \nu(\overline{D}) \geq \frac{p}{\alpha} \right\},$$

where $p \in (1, \alpha)$. Notice that in this case the set \mathcal{M}_p is closed, since \overline{D} is a closed set.

Here there are two cases. In case $p \in (1, e)$ the minimizing measure is given by

$$\mu_{Z_p}(z) = \mu_{Z_p}^\alpha(z) = \frac{1}{\alpha} \left[\mathbf{1}_{\{|z| \leq \sqrt{q}\}}(z) + \mathbf{1}_{\{\sqrt{p} \leq |z| \leq \sqrt{\alpha}\}}(z) \right] \cdot \frac{m(z)}{\pi} + \frac{p-q}{\alpha} m_{\{|z|=1\}},$$

where $q = q(p) < 1$ is defined as the solution of the equation $q(\log q - 1) = p(\log p - 1)$. In case $p \in [e, \alpha)$ the measure is given by

$$\mu_{Z_p}(z) = \frac{p}{\alpha} m_{\{|z|=1\}} + \frac{1}{\alpha} \mathbf{1}_{\{\sqrt{p} \leq |z| \leq \sqrt{\alpha}\}}(z) \cdot \frac{m(z)}{\pi}.$$

The proofs that these measures are the minimizers of the functional $I_\alpha(\nu)$ over the set \mathcal{M}_p are very similar to the case $p < 1$, and are left to the reader. A straightforward computation gives

$$I_\alpha(\mu_{Z_p}) = \begin{cases} \frac{\log \alpha}{2} - \frac{3}{4} + \frac{q^2(2 \log q - 1)}{4\alpha^2} - \frac{p^2(2 \log p - 1)}{4\alpha^2} & p \in (1, e); \\ \frac{\log \alpha}{2} - \frac{3}{4} - \frac{p^2(2 \log p - 1)}{4\alpha^2} & p \in [e, \alpha). \end{cases}$$

5.3. **‘Variational’ characterization of the minimizers.** The following simple results will be of use in Section 7.

Claim 8. Let $\mu, \nu \in \mathcal{M}_1(\mathbb{C})$ be probability measure with finite logarithmic energy and let $t \in [0, 1]$ be small. Then

$$\Sigma(t\nu + (1-t)\mu) = \Sigma(\mu) - 2t \left[\Sigma(\mu) - \int_{\mathbb{C}} U_\nu(w) d\mu(w) \right] + O(t^2).$$

Proof. From the definition of the logarithmic energy we have

$$\begin{aligned} \Sigma(t\nu + (1-t)\mu) &= t^2 \Sigma(\nu) + (1-t)^2 \Sigma(\mu) + 2t(1-t) \int_{\mathbb{C}} U_\nu(w) d\mu(w) \\ &= \Sigma(\mu) - 2t \left[\Sigma(\mu) - \int_{\mathbb{C}} U_\nu(w) d\mu(w) \right] + O(t^2). \end{aligned}$$

□

Claim 9. Let $\mu, \nu \in \mathcal{M}_1(\mathbb{C})$ and let $t \in [0, 1]$. Then

$$B_\alpha(t\nu + (1-t)\mu) \leq B_\alpha(\mu) + t[B_\alpha(\nu) - B_\alpha(\mu)].$$

Proof. By the definition of $B(\nu)$, and the linear properties of the logarithmic potential,

$$\begin{aligned} B_\alpha(t\nu + (1-t)\mu) &= 2 \sup_{w \in \mathbb{C}} \left\{ tU_\nu(w) + (1-t)U_\mu(w) - \frac{|w|^2}{2\alpha} \right\} \\ &\leq t \cdot 2 \sup_{w \in \mathbb{C}} \left\{ U_\nu(w) - \frac{|w|^2}{2\alpha} \right\} + (1-t) \cdot 2 \sup_{w \in \mathbb{C}} \left\{ U_\mu(w) - \frac{|w|^2}{2\alpha} \right\} \\ &= tB_\alpha(\nu) + (1-t)B_\alpha(\mu). \end{aligned}$$

□

In the following lemma we derive a characterization of the minimizers of the functional $I_\alpha(\nu)$.

Lemma 10. *Let $\mathcal{C} \subset \mathcal{M}_1(\mathbb{C})$ be a closed and convex set of measures. Suppose $\mu_{\min} \in \mathcal{C}$ is the unique minimizer of $I_\alpha(\nu)$ over the set \mathcal{C} . For $\mu \in \mathcal{C}$, we have that*

$$\int_{\mathbb{C}} U_\nu(w) d\mu(w) - \frac{B_\alpha(\nu)}{2} \leq \int_{\mathbb{C}} U_\mu(w) d\mu(w) - \frac{B_\alpha(\mu)}{2}, \quad \forall \nu \in \mathcal{C},$$

if and only if $\mu = \mu_{\min}$.

Proof. Suppose $\mu = \mu_{\min}$ and let $t > 0$ be small. For $\nu \in \mathcal{C}$, we have that $\mu_t = t\nu + (1-t)\mu_{\min} \in \mathcal{C}$. Therefore, using the previous claims

$$\begin{aligned} I_\alpha(\mu_{\min}) &\leq I_\alpha(\mu_t) \\ &= B_\alpha(t\nu + (1-t)\mu_{\min}) - \Sigma(t\nu + (1-t)\mu_{\min}) \\ &\leq B_\alpha(\mu_{\min}) + t[B_\alpha(\nu) - B_\alpha(\mu_{\min})] \\ &\quad - \Sigma(\mu_{\min}) + 2t \left[\Sigma(\mu_{\min}) - \int_{\mathbb{C}} U_\nu(w) d\mu_{\min}(w) \right] + O(t^2) \\ &= I_\alpha(\mu_{\min}) + t \left[B_\alpha(\nu) - B_\alpha(\mu_{\min}) + 2 \left\{ \Sigma(\mu_{\min}) - \int_{\mathbb{C}} U_\nu(w) d\mu_{\min}(w) \right\} \right] + O(t^2). \end{aligned}$$

Since $t > 0$ can be arbitrarily small, this implies

$$B_\alpha(\nu) - B_\alpha(\mu_{\min}) + 2 \left\{ \Sigma(\mu_{\min}) - \int_{\mathbb{C}} U_\nu(w) d\mu_{\min}(w) \right\} \geq 0.$$

In the other direction, assume

$$\int_{\mathbb{C}} U_\nu(w) d\mu(w) - \frac{B_\alpha(\nu)}{2} > \int_{\mathbb{C}} U_\mu(w) d\mu(w) - \frac{B_\alpha(\mu)}{2},$$

for some $\nu \in \mathcal{C}$. Using the argument above, we can find another measure μ_t such that $I_\alpha(\mu_t) < I_\alpha(\mu)$, thus μ is not the minimizer. □

Let $\mathcal{C} \subset \mathcal{M}_1(\mathbb{C})$ be a closed and convex set of measures, and let μ_{\min} be the measure that minimizes $I_\alpha(\nu)$ over all $\nu \in \mathcal{C}$. We will need the following simple bound.

Claim 10. For any $\nu \in \mathcal{C}$ we have

$$-\Sigma(\nu - \mu_{\min}) \leq I_\alpha(\nu) - I_\alpha(\mu_{\min}).$$

Proof. By the above lemma

$$\begin{aligned} \int_{\mathbb{C}} U_\nu(w) d\mu_{\min}(w) &\leq \int_{\mathbb{C}} U_{\mu_{\min}}(w) d\mu_{\min}(w) + \frac{B_\alpha(\nu)}{2} - \frac{B_\alpha(\mu_{\min})}{2} \\ &= \Sigma(\mu_{\min}) + \frac{B_\alpha(\nu)}{2} - \frac{B_\alpha(\mu_{\min})}{2}, \end{aligned}$$

which implies

$$\begin{aligned} -\Sigma(\nu - \mu_{\min}) &= -\Sigma(\nu) + 2 \int_{\mathbb{C}} U_\nu(w) d\mu_{\min}(w) - \Sigma(\mu_{\min}) \\ &\leq B_\alpha(\nu) - \Sigma(\nu) - [B_\alpha(\mu_{\min}) - \Sigma(\mu_{\min})] = I_\alpha(\nu) - I_\alpha(\mu_{\min}). \end{aligned}$$

□

6. PROBABILITY OF LARGE FLUCTUATIONS IN THE NUMBER OF ZEROS - LOWER BOUND

The goal of this section is to obtain the lower bound in Theorem 6, that is, we are looking for a lower bound for the probability $\mathbb{P}[n(r) = \lfloor pr^2 \rfloor]$, where $p \geq 0$, $p \neq 1$ is fixed. Let us write

$$k_0 = k_0(r, p) = \lfloor pr^2 \rfloor.$$

The main idea is to use Rouché's theorem. More precisely, we explicitly construct an event where the term $\left| \xi_{k_0} \frac{z^{k_0}}{\sqrt{k_0!}} \right|$ in the Taylor series of the GEF dominates the sum over all the other terms (on the circle $\{|z| = r\}$). This simple but effective method originally appeared in the paper [SoT2], and was later used in many other problems of this type.

6.1. Outline of the proof. We use the notation

$$b_k = b_k(r) = \frac{r^k}{\sqrt{k!}}, \quad k \in \mathbb{N},$$

and, using (3.1), we have the following bounds

$$(6.1) \quad \frac{1}{2k^{\frac{1}{4}}} \left(\frac{er^2}{k} \right)^{\frac{k}{2}} \leq b_k \leq \left(\frac{er^2}{k} \right)^{\frac{k}{2}}, \quad k \geq 1.$$

Let us consider the event $\{n(r) = k_0\}$. Rouché's theorem implies that

$$E_p = E_p(r) \stackrel{\text{def}}{=} \left\{ |\xi_{k_0}| b_{k_0} > \sum_{k \neq k_0} |\xi_k| b_k \right\} \subset \left\{ |\xi_{k_0}| \frac{r^{k_0}}{\sqrt{k_0!}} > \left| \sum_{k \neq k_0} \xi_k \frac{z^k}{\sqrt{k!}} \right| \right\} \subset \{n(r) = k_0\}.$$

We will construct an event that is contained in E_p , and thus obtain a lower bound for the probability of the event $\{n(r) = k_0\}$. Depending on p , we define an interval $I_p \subset \mathbb{R}^+$. We consider two main cases:

- Case 1. $0 \leq p < e$.
- In this case we define $q = q(p) \neq p$ to be the non-trivial solution of $q(\log q - 1) = p(\log p - 1)$.
 - We set $I_p = [p, q]$ in case $p < 1$, and $I_p = [q, p]$ in case $p > 1$.
- Case 2. $p \geq e$.
- In this case we set $I_p = [0, p]$.

In general, our strategy is to 'suppress' the terms b_k for which $\frac{k}{r^2} \in I_p$ (by choosing $|\xi_k|$ to be small), except for the main term b_{k_0} . We will also assume $|\xi_{k_0}| \geq 1$, which happens with a constant probability. By (6.1), this implies

$$(6.2) \quad \begin{aligned} |\xi_{k_0}| b_{k_0} &\geq \exp \left(\frac{p}{2} \log \left(\frac{e}{p} \right) r^2 - C \log(pr^2) \right) \\ &\geq \exp \left(\frac{p}{2} \log \left(\frac{e}{p} \right) r^2 - C \log r \right), \end{aligned}$$

for r sufficiently large (or $|\xi_{k_0}| b_{k_0} \geq 1$ in case $p = k_0 = 0$).

6.1.1. *Sketch of the proof in case $p < 1$.* We now explain the idea of the proof in Case 1 (the proof of the other case is similar). Using bounds for the factorial, we find

$$(6.3) \quad \frac{b_{k_0}}{b_k} = \exp \left(\frac{p}{2} \log \left(\frac{e}{p} \right) r^2 - \frac{k}{2} \log \left(\frac{er^2}{k} \right) + \text{error terms} \right).$$

Put $k_1 = \lfloor qr^2 \rfloor + 1$, where $q = q(p)$ as defined above. The estimate (6.3) implies that $\frac{b_{k_0}}{b_k}$ is small for k not in the range $\{k_0, \dots, k_1\}$. That is, the tail

$$\left| \sum_{k \notin \{k_0, \dots, k_1\}} \xi_k \frac{z^k}{\sqrt{k!}} \right| \leq \sum_{k \notin \{k_0, \dots, k_1\}} |\xi_k| b_k,$$

is small compared to $|\xi_{k_0}| b_{k_0}$, with sufficiently large probability. To make the sum over $k \in \{k_0 + 1, \dots, k_1\}$ small, we consider the event where a $|\xi_k|$ is at most $\left(\frac{b_{k_0}}{b_k}\right)^{-1}$, for k in this range. The probability of this event is

$$\exp \left(-2 \cdot \sum_{k \in \{k_0 + 1, \dots, k_1\}} \log \left(\frac{b_{k_0}}{b_k} \right) + \text{error terms} \right).$$

We obtain the lower bound in Theorem 6, after verifying

$$2 \cdot \sum_{k \in \{k_0 + 1, \dots, k_1\}} \log \left(\frac{b_{k_0}}{b_k} \right) = -Z_p r^4 + \text{error terms},$$

where $Z_p = \int_p^q x \log x \, dx$.

6.2. **The main terms.** Consider now $p \geq 0$, $p \neq 1$. We define the set of main terms by

$$M = \left\{ k \in \mathbb{N} : \frac{k}{r^2} \in I_p \right\}.$$

For $k \in \mathbb{N}^+$, let us define $A_{p,k} = \frac{b_{k_0}}{b_k}$. From (6.1), we find the following bounds

$$-C \log(k_0 + 1) \leq \log A_{p,k} - \left[\frac{p}{2} \log \left(\frac{e}{p} \right) r^2 - \frac{k}{2} \log \left(\frac{er^2}{k} \right) \right] \leq C \log(k + 1),$$

thus, for r sufficiently large,

$$(6.4) \quad \left| \log A_{p,k} - \left[\frac{p}{2} \log \left(\frac{e}{p} \right) r^2 - \frac{k}{2} \log \left(\frac{er^2}{k} \right) \right] \right| \leq C_1 \log(k + 1),$$

for some numerical constant $C_1 > 0$. Notice that if $k = \alpha r^2$ for some $\alpha \geq 0$, then

$$B_\alpha(r) \stackrel{\text{def}}{=} \frac{p}{2} \log \left(\frac{e}{p} \right) r^2 - \frac{k}{2} \log \left(\frac{er^2}{k} \right) = \left[p \log \left(\frac{e}{p} \right) - \alpha \log \left(\frac{e}{\alpha} \right) \right] \frac{r^2}{2} = [p(1 - \log p) - \alpha(1 - \log \alpha)] \frac{r^2}{2}.$$

This means that M contains the terms for which $B_\alpha(r) \leq 0$. Since $A_{p,k} \cdot \exp(-2C_1 \log(k + 1)) \leq 1$ for $k \in M$, we have

$$\begin{aligned}
\mathbb{P} \left[|\xi_k| \leq \frac{1}{6r^2 \cdot (k+1)^{2C_1}} \cdot A_{p,k} \right] &\geq \frac{C}{r^4 (k+1)^{4C_1}} A_{p,k}^2 \\
&\geq \exp \left(p \log \left(\frac{e}{p} \right) r^2 - k \log \left(\frac{er^2}{k} \right) - C \log((p+1)r) \right) \\
&\geq \exp \left(p \log \left(\frac{e}{p} \right) r^2 - k \log \left(\frac{er^2}{k} \right) - C \log r \right),
\end{aligned}$$

for r sufficiently large. We notice that in Case 1 there at most $3r^2$ elements in M . We introduce the following event:

$$E_M^1 = \left\{ |\xi_k| \leq \frac{1}{6r^2 (k+1)^{2C_1}} \cdot A_{p,k}, \quad \text{for all } k \in M \right\}.$$

Clearly on the event $\{|\xi_0| \geq 1\} \cap E_M^1$, we have

$$\sum_{k \in M} |\xi_k| b_k \leq \sum_{k \in M} \frac{A_{p,k}}{6r^2} \cdot b_k \leq \frac{1}{2} b_{k_0} \leq \frac{1}{2} |\xi_{k_0}| b_{k_0}.$$

On the other hand,

$$\begin{aligned}
\mathbb{P}[E_M^1] &= \prod_{k \in M} \mathbb{P} \left[|\xi_k| \leq \frac{1}{6r^2 \cdot (k+1)^{2C_1}} \cdot A_{p,k} \right] \\
&\geq \exp \left(\sum_{k \in M} \left[p \log \left(\frac{e}{p} \right) r^2 - k \log \left(\frac{er^2}{k} \right) - C \log r \right] \right) \\
&\geq \exp \left(\sum_{k \in M} \left[p \log \left(\frac{e}{p} \right) r^2 - k \log \left(\frac{er^2}{k} \right) \right] + O(r^2 \log r) \right).
\end{aligned}$$

In Section 6.4 we show

$$\sum_{k \in M} \left[p \log \left(\frac{e}{p} \right) r^2 - k \log \left(\frac{er^2}{k} \right) \right] = -Z_p \cdot r^4 + O(r^2 \log^2 r).$$

Case 2 is similar, and now there are at most $\lfloor 2pr^2 \rfloor$ elements in M . Consider the event:

$$E_M^2 = \left\{ |\xi_k| \leq \frac{1}{4pr^2 (k+1)^{2C_1}} \cdot A_{p,k}, \quad \text{for all } k \in M \right\}.$$

On the event $\{|\xi_0| \geq 1\} \cap E_M^2$, we have

$$\sum_{k \in M} |\xi_k| b_k \leq \sum_{k \in M} \frac{A_{p,k}}{4pr^2 (k+1)^{2C_1}} \cdot b_k \leq \frac{1}{2} b_{k_0} \leq \frac{1}{2} |\xi_{k_0}| b_{k_0}.$$

We also have, for r sufficiently large,

$$\begin{aligned} \mathbb{P}[E_M^2] &= \prod_{k \in M} \mathbb{P}\left[|\xi_k| \leq \frac{1}{4pr^2(k+1)^{2C_1}} \cdot A_{p,k}\right] \\ &\geq \exp\left(\sum_{k \in M} \left[p \log\left(\frac{e}{p}\right) r^2 - k \log\left(\frac{er^2}{k}\right) - C \log r\right]\right) \\ &\geq \exp\left(\sum_{k \in M} \left[p \log\left(\frac{e}{p}\right) r^2 - k \log\left(\frac{er^2}{k}\right)\right] + O(r^2 \log^2 r)\right). \end{aligned}$$

6.3. Tail bounds. For technical reasons we consider two parts of the ‘tail’ separately, the far tail and the close tail.

6.3.1. The far tail. Let $r > 0$ be sufficiently large, $\alpha > 10$, and $N = \lfloor \alpha r^2 \rfloor + 1$. Recall the tail of the GEF is given by

$$T_N(z) = \sum_{k=N+1}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}, \quad z \in \mathbb{C}.$$

By Lemma 3 we have, outside an exceptional event E_T of probability at most $\exp(-Cr^6)$,

$$|T_N(z)| \leq \exp\left(\frac{N}{2} \log\left(\frac{4}{\alpha}\right)\right), \quad |z| \leq r.$$

We will take

$$\alpha = \alpha(p) = \begin{cases} 16 & 0 \leq p \leq 11; \\ 5+p & p > 11. \end{cases}$$

If $p \leq 11$, then $\frac{p}{2} \log\left(\frac{e}{p}\right) > -8$, and therefore by (6.2) we have $|T_N(z)| \leq \exp(-8r^2) < \frac{1}{4} |\xi_{k_0}| b_{k_0}$, when r is sufficiently large. Similarly for $p > 11$, we have $\frac{p}{2} \log\left(\frac{e}{p}\right) > \frac{5+p}{2} \log\left(\frac{4}{5+p}\right)$, and

$$|T_N(z)| \leq \exp\left(\frac{5+p}{2} \log\left(\frac{4}{5+p}\right) \cdot r^2\right) < \frac{1}{4} |\xi_{k_0}| b_{k_0}.$$

6.3.2. The close tail. We now consider all the terms such that $\frac{k}{r^2} \notin I_p$, but $k \leq N$. For these terms we have $\frac{k}{2} \log\left(\frac{er^2}{k}\right) \leq \frac{p}{2} \log\left(\frac{e}{p}\right) r^2$, and the lower bound of (6.4) implies $b_k \leq b_{k_0} (k+1)^{C_1}$. In Case 1, we notice there are at most $\lfloor 17r^2 \rfloor$ elements in the close tail, let us denote their indices by M' . Introduce the following event:

$$E_{M'}^1 = \left\{ |\xi_k| \leq \frac{1}{70r^2(k+1)^{C_1}}, \quad \text{for all } k \in M' \right\}.$$

On the event $\{|\xi_0| \geq 1\} \cap E_{M'}^1$ we have

$$\sum_{k \in M'} |\xi_k| b_k \leq \sum_{k \in M'} \frac{1}{70r^2(k+1)^{C_1}} \cdot b_k < \frac{1}{4} b_{k_0} \leq \frac{1}{4} |\xi_0| b_{k_0}.$$

In addition,

$$\mathbb{P} [E_{M'}^1] \geq \prod_{k=0}^{\lfloor 17r^2 \rfloor} \frac{1}{2} \cdot \left(\frac{1}{70r^2 (k+1)^{C_1}} \right)^2 \geq (Cr^{4+2C_1})^{-Cr^2} \geq \exp(-Cr^2 \log r).$$

Similarly, in Case 2, there are at most $\lfloor 6r^2 \rfloor$ elements in the close tail. Let us again denote their indices by M' . Consider the event:

$$E_{M'}^2 = \left\{ |\xi_k| \leq \frac{1}{24r^2 (k+1)^{C_1}}, \quad \text{for all } k \in M' \right\}.$$

On the event $\{|\xi_0| \geq 1\} \cap E_{M'}^2$ we have

$$\sum_{k \in M'} |\xi_k| b_k \leq \sum_{k \in M'} \frac{1}{24r^2 (k+1)^{C_1}} \cdot b_k < \frac{1}{4} b_{k_0} \leq \frac{1}{4} |\xi_0| b_{k_0}.$$

Now, for $k \in M'$ we have $k \leq r^3$ (for r sufficiently large), and therefore

$$\begin{aligned} \mathbb{P} [E_{M'}^2] &\geq \prod_{k \in M'} \frac{1}{2} \cdot \left(\frac{1}{24r^2 (k+1)^{C_1}} \right)^2 \geq \prod_{k=0}^{\lfloor 6r^2 \rfloor} Cr^{-2(2+3C_1)} \\ &\geq \exp(-Cr^2 \log r). \end{aligned}$$

6.4. Combining the estimates and finishing the proof. Notice that the events $\{|\xi_0| \geq 1\}$, E_T , E_M^j , and $E_{M'}^j$ are all independent ($j \in \{1, 2\}$). Therefore the probability of the event $|\xi_{k_0} b_{k_0}| > \sum_{k \neq k_0} |\xi_k| b_k$ is at least

$$\exp \left(\sum_{k \in M} \left[p \log \left(\frac{e}{p} \right) r^2 - k \log \left(\frac{er^2}{k} \right) \right] + O(r^2 \log^2 r) \right).$$

Consider now the function

$$l(x) = \left[p \log \left(\frac{e}{p} \right) - x \log \left(\frac{e}{x} \right) \right] r^2, \quad x \geq 0.$$

This function has a single minimum at $x = 1$. Therefore,

$$\sum_{k \in M} \left[p \log \left(\frac{e}{p} \right) r^2 - k \log \left(\frac{er^2}{k} \right) \right] = \sum_{k \in M} l \left(\frac{k}{r^2} \right) = - \left| \int_p^{q(p)} l(x) dx \right| \cdot r^2 + O(C(p) r^2),$$

where $C(p) = \max \{ p \log \left(\frac{p}{e} \right), 1 \}$ (in Case 2 we take $q(p) = 0$). Hence, we obtained the lower bound,

$$\mathbb{P} [n(r) = \lfloor pr^2 \rfloor] \geq \exp \left(- \left| \int_p^{q(p)} \left[p \log \left(\frac{e}{p} \right) - x \log \left(\frac{e}{x} \right) \right] dx \right| \cdot r^4 + O(r^2 \log^2 r) \right),$$

for r sufficiently large.

To prove Theorem (6) it remains to verify the identity

$$\left| \int_p^{q(p)} \left[p \log \left(\frac{e}{p} \right) - x \log \left(\frac{e}{x} \right) \right] dx \right| = \left| \int_p^{q(p)} x \log x dx \right|.$$

Let us consider Case 1, for $p < 1$. Set $t(x) = x \log\left(\frac{e}{x}\right)$ and $q = q(p)$. We have

$$\begin{aligned}
 \int_p^q \left[p \log\left(\frac{e}{p}\right) - x \log\left(\frac{e}{x}\right) \right] dx &= (q-p) p \log\left(\frac{e}{p}\right) - \int_p^q t(x) dx \\
 &= (q-p) p \log\left(\frac{e}{p}\right) - xt(x)|_{x=p}^q + \int_p^q xt'(x) dx \\
 &= (q-p) p \log\left(\frac{e}{p}\right) - qt(q) + pt(p) - \int_p^q x \log x dx \\
 &= q \left(p \log\left(\frac{e}{p}\right) - q \log\left(\frac{e}{q}\right) \right) - \int_p^q x \log x dx \\
 &= - \int_p^q x \log x dx,
 \end{aligned}$$

where in the last line we used the definition of $q = q(p)$. The other cases are proved in a similar way.

7. THE CONDITIONAL DISTRIBUTION OF THE ZEROS

In this section we describe the distribution of the zeros of the GEF, conditioned on a prescribed number of zeros inside the disk $\{|z| \leq r\}$ (recall that we denote this number by $n(r)$). We will again consider two main cases:

- Case 1. Deficiency in the number of zeros: $F(p; r) = \{n(r) \leq pr^2\}$, where $p \in [0, 1)$.
 - In particular, this implies the case $p = 0$ of Theorem 1.
- Case 2. Overcrowding of zeros: $M(p; r) = \{n(r) \geq pr^2\}$, where $p > 1$.
 - We consider separately the range $p \in (1, e)$, and the range $p \in [e, \infty)$.

For each of the cases we define the limiting conditional distribution, by the following Radon measures:

$$d\mu_{Z_p}^{\mathbb{C}}(z) = \begin{cases} \left[\mathbf{1}_{\{0 \leq |w| \leq \sqrt{p}\}}(z) + \mathbf{1}_{\{\sqrt{q} \leq |w|\}}(z) \right] \cdot \frac{dm(z)}{\pi} + (q-p) dm_{\{|z|=1\}} & p \in [0, 1); \\ \left[\mathbf{1}_{\{0 \leq |w| \leq \sqrt{q}\}}(z) + \mathbf{1}_{\{\sqrt{p} \leq |w|\}}(z) \right] \cdot \frac{dm(z)}{\pi} + (p-q) dm_{\{|z|=1\}} & p \in (1, e); \\ \mathbf{1}_{\{\sqrt{p} \leq |w|\}}(z) \cdot \frac{dm(z)}{\pi} + p \cdot dm_{\{|z|=1\}} & p \geq e. \end{cases}$$

Here $m_{\{|z|=1\}}$ is Lebesgue measure on the circle $|z| = 1$, normalized to be a probability measure and $\mathbf{1}_A(z)$ is the indicator of the set A . We recall that $q = q(p)$ was defined before the statement of Theorem 6.

Suppose now $\varphi \in C_0^2(\mathbb{C})$ is a test function, which is twice continuously differentiable and with compact support, and recall

$$\mathfrak{D}(\varphi) = \|\nabla \varphi\|_{L^2(m)}^2 = \int_{\mathbb{C}} (\varphi_x^2 + \varphi_y^2) dm(z).$$

Let $n_{F_{\mathbb{C}}}(\varphi; r)$ be the linear statistics associated with φ , recall that $\mathbb{E}[n_{F_{\mathbb{C}}}(\varphi; r)] = r^2 \cdot \frac{1}{\pi} \int_{\mathbb{C}} \varphi(w) dm(w)$, and consider the event

$$L(p, \varphi, \lambda) = L(p, \varphi, \lambda; r) = \left\{ \left| n_{F_{\mathbb{C}}}(\varphi; r) - r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w) \right| \geq \lambda \right\}.$$

The following theorem shows that conditioned on the event $F(p; r)$ (or $M(p; r)$), the value of $n_{F_{\mathbb{C}}}(\varphi; r)$ is unlikely to be far from $r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w)$. Recall that by $\mathbb{E}_{F(p; r)}[\cdot]$ (resp. $\mathbb{P}_{F(p; r)}[\cdot]$) we denote the conditional expectation (resp. probability) on the event $F(p; r)$.

Theorem 8. Suppose $C' > 0$ is fixed. For $\lambda \in (0, C'r^2)$ and r sufficiently large we have,

$$\mathbb{P}_{F(p;r)} [L(p, \varphi, \lambda; r)], \mathbb{P}_{M(p;r)} [L(p, \varphi, \lambda; r)] \leq \exp \left(-\frac{C_p}{\mathfrak{D}(\varphi)} \cdot \lambda^2 + C_\varphi r^2 \log^2 r \right),$$

where $C_\varphi > 0$ is some constant that depends on $\omega(\varphi, t)$ - the modulus of continuity of the function φ , and $C_p > 0$ is a constant depending only on p (and which can be replaced by an absolute constant for $p \leq e$).

Remark 7. It will be clear from the proof, that we can take the test function φ depending on r , such that its modulus of continuity satisfies $\omega(\varphi, t) = O(r^{C_3} t^{C_4})$, for some numerical constants $C_3, C_4 > 0$. In that case, the constant C_φ will depend only on C_3 and C_4 .

This theorem implies the convergence in distribution of the zero counting measure, conditioned on the event $F(p)$ (or $M(p)$). We denote by \mathcal{Z}_r^p the zero set of $F_{\mathbb{C}}$ conditioned on the occurrence of the event $F(p; r)$, and write $[\mathcal{Z}_r^p]$ for the corresponding counting measure (similar definitions can be made for the event $M(p)$).

Theorem 9. Let $\varphi \in C_0^2(\mathbb{C})$ be a fixed test function. As $r \rightarrow \infty$,

$$\mathbb{E}_{F(p;r)} [n_{F_{\mathbb{C}}}(\varphi; r)], \mathbb{E}_{M(p;r)} [n_{F_{\mathbb{C}}}(\varphi; r)] = r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{\mathcal{Z}_p}^{\mathbb{C}}(w) + O(r \log^2 r).$$

In addition, as $r \rightarrow \infty$, the scaled zero counting measure $\frac{1}{r^2} [\mathcal{Z}_r^p] \left(\frac{\cdot}{r} \right) \rightarrow \mu_{\mathcal{Z}_p}^{\mathbb{C}}$ in distribution, where the convergence is in the vague topology. That is, for any continuous test function ϕ with compact support, we have

$$\frac{1}{r^2} \int_{\mathbb{C}} \phi d[\mathcal{Z}_r^p] \left(\frac{\cdot}{r} \right) = \frac{1}{r^2} \sum_{z \in \mathcal{Z}_r^p} \phi \left(\frac{z}{r} \right) \xrightarrow[r \rightarrow \infty]{d} \int_{\mathbb{C}} \phi(w) d\mu_{\mathcal{Z}_p}^{\mathbb{C}}(w).$$

An analogous result holds for the event $M(p; r)$.

7.1. Preliminaries. Notice that for any $p \geq 0$, if α is sufficiently large, then $\mu_{\mathcal{Z}_p}^{\mathbb{C}}(D(0, \sqrt{\alpha})) = \alpha$. We are going to work with the following probability measures, which are the normalized truncations of $\mu_{\mathcal{Z}_p}^{\mathbb{C}}$:

$$d\mu_{\mathcal{Z}_p}^{\alpha}(z) = \begin{cases} \frac{1}{\alpha} \left[\mathbf{1}_{\{|z| \leq \sqrt{p}\}}(z) + \mathbf{1}_{\{\sqrt{q} \leq |z| \leq \sqrt{\alpha}\}}(z) \right] \cdot \frac{dm(z)}{\pi} + \frac{q-p}{\alpha} dm_{\{|z|=1\}} & p \in [0, 1); \\ \frac{1}{\alpha} \left[\mathbf{1}_{\{|z| \leq \sqrt{q}\}}(z) + \mathbf{1}_{\{\sqrt{p} \leq |z| \leq \sqrt{\alpha}\}}(z) \right] \cdot \frac{dm(z)}{\pi} + \frac{p-q}{\alpha} dm_{\{|z|=1\}} & p \in (1, e); \\ \frac{1}{\alpha} \mathbf{1}_{\{\sqrt{p} \leq |z| \leq \sqrt{\alpha}\}}(z) \cdot \frac{dm(z)}{\pi} + \frac{p}{\alpha} dm_{\{|z|=1\}} & p \in [e, \alpha). \end{cases}$$

In Section 5, we showed that these measures are the minimizers of the functional $I(\nu) = I_{\alpha}(\nu)$ over the sets $\mathcal{F}_p = \{\nu \in \mathcal{M}_1(\mathbb{C}) : \nu(D) \leq \frac{p}{\alpha}\}$, where $p < 1$, and $\mathcal{M}_p = \{\nu \in \mathcal{M}_1(\mathbb{C}) : \nu(\overline{D}) \geq \frac{p}{\alpha}\}$, where $p > 1$.

We introduce the following notation

$$\mathcal{L}_{\varphi, \tau, \lambda} \stackrel{\text{def}}{=} \left\{ \nu \in \mathcal{M}_1(\mathbb{C}) : \left| \int_{\mathbb{C}} \varphi(w) d\nu(w) - \tau \right| \geq \lambda \right\}.$$

The key tool that we will use is the next claim, which can be seen as an effective form of the fact $I(\nu)$ is strictly convex. It shows that if a measure $\nu \in \mathcal{F}_p$ (or \mathcal{M}_p) is far from the minimizer $\mu_{\mathcal{Z}_p}^{\alpha}$ (with respect to the test function φ), then $I(\nu)$ is relatively large.

Claim 11. Let $\tau = \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^\alpha(w)$. For any compactly supported measure $\nu \in \mathcal{F}_p \cap \mathcal{L}_{\varphi, \tau, \lambda}$, we have

$$I(\nu) - I(\mu_{Z_p}^\alpha) \geq \frac{2\pi}{\mathfrak{D}(\varphi)} \cdot \lambda^2.$$

The same result holds if we replace \mathcal{F}_p by \mathcal{M}_p .

Proof. In Section 5, we prove that the measure $\mu_{Z_p}^\alpha$ minimizes $I(\nu)$ over the set \mathcal{F}_p . Now, combining Lemma 9 and Claim 10 we have for $\nu \in \mathcal{F}_p$

$$\begin{aligned} \lambda &\leq \left| \int_{\mathbb{C}} \varphi(w) d\nu(w) - \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^\alpha(w) \right| \leq \frac{1}{\sqrt{2\pi}} \sqrt{\mathfrak{D}(\varphi)} \sqrt{-\Sigma(\nu - \mu_{Z_p}^\alpha)}, \\ &\leq \frac{1}{\sqrt{2\pi}} \sqrt{\mathfrak{D}(\varphi)} \sqrt{I(\nu) - I(\mu_{Z_p}^\alpha)}. \end{aligned}$$

The same proof applies for \mathcal{M}_p as well. \square

7.2. Truncation of the power series and estimates for the joint distribution of the zeros.

We start by recalling some of the results we proved in Section 4. Let $r > 0$ be sufficiently large. Put $\alpha = Nr^{-2}$, $\lambda = \log r$, $t = \gamma = r^{-C_2}$, with $C_2 \geq 4$, and $N_0 = \lfloor \lambda r^2 \rfloor + 1$, $N_1 = \lfloor 2\lambda r^2 \rfloor + 1$. Let φ be a test function supported on the disk $D(0, B)$, with fixed $B \geq 1$. We found that there exist events E_{reg} and E_{reg}^N , $N \in \{N_0, \dots, N_1\}$, such that

$$E_{\text{reg}} = \bigcup_{N=N_0}^{N_1} E_{\text{reg}}^N, \quad E_{\text{reg}}^c \text{ is negligible.}$$

Remark 8. To find the event E_{reg} we applied Lemma 7 and Lemma 8. We may choose the parameter A in Lemma 7 to be arbitrarily large (but fixed). For r sufficiently large this gives,

$$\mathbb{P}[E_{\text{reg}}^c] \leq \exp(-C_A r^4),$$

where C_A is a constant depending on A (and B), such that $C_A \rightarrow \infty$ as $A \rightarrow \infty$.

If we introduce the scaled polynomial

$$P_{N,L}(z) = \sum_{k=0}^N \xi_k \frac{(Lz)^k}{\sqrt{k!}}, \quad z \in \mathbb{C},$$

then, on the event E_{reg}^N , we have

$$(7.1) \quad n_{P_{N,L}}\left(\frac{r-K_0}{L}\right) \leq n(r) \leq n_{P_{N,L}}\left(\frac{r+K_0}{L}\right),$$

$$(7.2) \quad \left| n_{F_{\mathbb{C}}}(\varphi; r) - n_{P_{N,L}}\left(\varphi; \frac{r}{L}\right) \right| \leq CM_0 \cdot \omega(\varphi; K_0 r^{-1}),$$

where $M_0 = 8B^3 r^3$, and $K_0 = 2M_0 \gamma \leq Cr^{3-C_2} = O(\frac{1}{r})$. We choose the parameter L as follows:

$$(7.3) \quad L = L(r; B) = \begin{cases} (1+t)^{-1}(r-K_0) & \text{in Case 1;} \\ (1-t)^{-1}(r+K_0) & \text{in Case 2.} \end{cases}$$

Now, for $\tau \in \mathbb{R}$, $\eta \geq 0$, define the following set

$$L_{\varphi, \tau, \eta}^N = L_{\varphi, \tau, \eta}^N(t) = \left\{ z : \left| \frac{1}{N} \sum_{j=1}^N \varphi(z_j) - \tau \right| \geq \eta + \omega(\varphi; t) \right\}.$$

Put $Z_N = \{n_{P_{N,L}}(1+t) \leq pr^2\} \cap L_{\varphi, \tau, \eta}^N \subset \mathbb{C}^N$. We showed in Section 4.3 that

$$(7.4) \quad \left\{ \mu_z^t : z \in Z_N \right\} \subset \left\{ \mu_z^t : \mu_z^t(D) \leq \frac{p}{\alpha} \text{ and } z \in L_{\varphi, \tau, \lambda}^N \right\} \subset \mathcal{F}_p \cap \mathcal{L}_{\varphi, \tau, \eta} \stackrel{\text{def}}{=} \mathcal{Z},$$

where we used (4.11). The bound (4.8) gives

$$(7.5) \quad \begin{aligned} & \log \mathbb{P} \left[\{n_{P_{N,L}}(1+t) \leq pr^2\} \cap L_{\varphi, \tau, \eta}^N \cap E_{\text{reg}}^N \right] \\ & \leq -N^2 \left[\inf_{\nu \in \mathcal{Z}} I_\alpha(\nu) - \frac{1}{2} \log \left(\frac{N}{L^2} \right) + \frac{3}{4} \right] + L^2 \log L \cdot O \left(\log L + \log \frac{1}{t} + tL^2 \sqrt{\log L} \right) \\ & = -N^2 \left[\inf_{\nu \in \mathcal{Z}} I_\alpha(\nu) - \frac{1}{2} \log \alpha + \frac{3}{4} \right] + O(r^2 \log^2 r). \end{aligned}$$

We can bound the probability of the event $\mathbb{P}[\{n_{P_{N,L}}(1+t) \geq pr^2\} \cap L_{\varphi, \tau, \eta}^N \cap E_{\text{reg}}^N]$ in a similar way.

7.3. Large fluctuations in the number of zeros and linear statistics. Let $\kappa \geq 0$ be sufficiently large (depending on r). Recall the event

$$F(p) \cap L(p, \varphi, \kappa) = F(p; r) \cap L(p, \varphi, \kappa; r) = \{n_{F_C}(r) \leq pr^2\} \cap \left\{ \left| n_{F_C}(\varphi; r) - r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w) \right| \geq \kappa \right\}.$$

By the definition of the measures $\mu_{Z_p}^\alpha$ and $\mu_{Z_p}^{\mathbb{C}}$, we have for α sufficiently large (depending on p and B)

$$r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w) = r^2 \cdot \alpha \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^\alpha(w) = N \cdot \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^\alpha(w).$$

In addition, using (7.2), we get

$$\sum_{z \in \mathcal{Z}(P_{N,L})} \varphi \left(z \cdot \left(\frac{r}{L} \right)^{-1} \right) = n_{P_{N,L}} \left(\varphi; \frac{r}{L} \right) = n_{F_C}(\varphi; r) + O(M_0 \cdot \omega(\varphi; Kr^{-1})),$$

where $\mathcal{Z}(P_{N,L}) = \{z_1, \dots, z_N\}$ is the zero set of the polynomial $P_{N,L}$. Since $\frac{r}{L} = 1 + O(r^{2-C_2})$ by (7.3), and because φ is supported inside $D(0, B)$, we obtain

$$\begin{aligned} \sum_{j=1}^N \varphi(z_j) &= \sum_{j=1}^N \varphi \left(z_j \cdot \left(\frac{r}{L} \right)^{-1} \right) + O(N \cdot B \cdot r^{2-C_2}) \\ &= n_{F_C}(\varphi; r) + O_B(r^{4-C_2} \log r + r^3 \cdot \omega(\varphi; Cr^{2-C_2})) \\ &= n_{F_C}(\varphi; r) + O_B(N \cdot E_1(\varphi; r)), \end{aligned}$$

where $N \cdot E_1(\varphi; r) = r^3 (r^{2-C_2} + \omega(\varphi; Cr^{2-C_2}))$, using $N = O(r^2 \log r)$.

Remark 9. If the test function φ depends on r in such a way that its modulus satisfies $\omega(\varphi, t) = O(r^{C_3} t^{C_4})$, for some numerical constants $C_3, C_4 > 0$, then we can choose C_2 sufficiently large to make $C_B \cdot N \cdot E_1(\varphi; r) \leq C$, for r sufficiently large.

We conclude that on the event $L(p, \varphi, \kappa; r) \cap E_{\text{reg}}^N$, we have

$$(7.6) \quad \left| \frac{1}{N} \sum_{j=1}^N \varphi(z_j) - \int_{\mathbb{C}} \varphi(w) \, d\mu_{Z_p}^\alpha(w) \right| \geq \frac{\kappa}{N} - C_B \cdot E_1(\varphi; r) \stackrel{\text{def}}{=} \frac{\kappa_1}{N},$$

with some constant $C_B \geq 1$, depending only on B . Let $\tau = \int_{\mathbb{C}} \varphi(w) \, d\mu_{Z_p}^\alpha(w)$ and assume that κ is sufficiently large so that $\kappa_1 > 0$. Claim 11 (with $\lambda = \frac{\kappa_1}{N}$) implies that

$$I_\alpha(\nu) \geq I_\alpha(\mu_{Z_p}^\alpha) + \frac{C}{\mathfrak{D}(\varphi)} \cdot \left(\frac{\kappa_1}{N}\right)^2, \quad \nu \in \mathcal{Z}.$$

By (7.5), (7.6), we have

$$(7.7) \quad \begin{aligned} \log \mathbb{P}[F(p; r) \cap L(p, \varphi, \kappa; r) \cap E_{\text{reg}}^N] &\leq \log \mathbb{P}[\{n_{P_N, L}(1+t) \leq pr^2\} \cap L_{\varphi, \tau, \kappa_1}^N \cap E_{\text{reg}}^N] \\ &\leq -N^2 \left[I_\alpha(\mu_{Z_p}^\alpha) + \frac{C}{\mathfrak{D}(\varphi)} \cdot \left(\frac{\kappa_1}{N}\right)^2 - \frac{1}{2} \log \alpha + \frac{3}{4} \right] + O(r^2 \log^2 r) \\ &\leq \log \mathbb{P}[F(p; r)] - \frac{C}{\mathfrak{D}(\varphi)} \cdot \kappa_1^2 + O(r^2 \log^2 r), \end{aligned}$$

where we used in the first inequality $\frac{\kappa}{N} \geq \frac{\kappa_1}{N} + \omega(\varphi; t)$ (since $\omega(\varphi; t) \leq E_1(\varphi; r)$), and in the third inequality

$$-N^2 \left[I_\alpha(\mu_{Z_p}^\alpha) - \frac{1}{2} \log \alpha + \frac{3}{4} \right] = -Z_p r^4 \leq \log \mathbb{P}[F(p; r)] + O(r^2 \log^2 r).$$

7.3.1. *Finishing the proof of Theorem 8.* Rewriting (7.7) we find that

$$\mathbb{P}[F(p) \cap L(p, \varphi, \kappa) \cap E_{\text{reg}}^N] \leq \mathbb{P}[F(p)] \exp \left(-\frac{C}{\mathfrak{D}(\varphi)} \cdot \kappa_1^2 + O(r^2 \log^2 r) \right),$$

with $\kappa_1 = \kappa - C_B N \cdot E_1(\varphi; r)$. By Remark (9), if $\omega(\varphi, t) = O(r^{C_3} t^{C_4})$ with some constants $C_3, C_4 > 0$, we can choose C_2 sufficiently large, so that $\kappa_1 \geq \kappa - C$, if r is sufficiently large. We thus have,

$$\begin{aligned} \mathbb{P}[F(p) \cap L(p, \varphi, \kappa)] &\leq \mathbb{P}[E_{\text{reg}}^c] + \sum_{N=N_0}^{N_1} \mathbb{P}[F(p) \cap L(p, \varphi, \kappa) \cap E_{\text{reg}}^N] \\ &\leq \mathbb{P}[E_{\text{reg}}^c] + (N_1 - N_0 + 1) \mathbb{P}[F(p)] \exp \left(-\frac{C}{\mathfrak{D}(\varphi)} \cdot \kappa_1^2 + O(r^2 \log^2 r) \right) \\ &\leq \mathbb{P}[E_{\text{reg}}^c] + \mathbb{P}[F(p)] \exp \left(-\frac{C}{\mathfrak{D}(\varphi)} \cdot \kappa^2 + \frac{C\kappa}{\mathfrak{D}(\varphi)} + O(r^2 \log^2 r) \right). \end{aligned}$$

We can assume w.l.o.g. that $\sup\{\varphi(w) : w \in \mathbb{C}\} = 1$, and that $\mathfrak{D}(\varphi) > c$ for some constant $c = c(B) > 0$ (since φ is supported on $D(0, B)$). In addition, by Remark 8, we can choose $C_5 > 0$ as large as we wish (but fixed). such that $\mathbb{P}[E_{\text{reg}}^c] \leq \exp(-C_5 r^4)$ for r sufficiently large. Finally, we conclude that for $\kappa \leq Cr^2$, we have

$$\mathbb{P}[F(p) \cap L(p, \varphi, \kappa)] \leq \mathbb{P}[F(p)] \exp \left(-\frac{C}{\mathfrak{D}(\varphi)} \cdot \kappa^2 + O(r^2 \log^2 r) \right).$$

This completes the proof of Theorem 8 in the case $p \in [0, 1)$. The proofs of the other cases go along the same lines, and we leave them to the reader. We note that in the case $p \geq e$, the constant in the statement of the theorem may depend on p .

Remark 10. Recall that $F(0) = F(0; r) = H_r$ is the hole event for $\{|z| < r\}$. Let $\varepsilon \in (r^{-2}, 1)$ and $\gamma \in (1, 2]$. Theorem 2 in Section 1 follows from Theorem 8 by considering a positive (say radial) test function $\varphi = \varphi_\varepsilon$, such that

$$\varphi(z) = \varphi(|z|) = \begin{cases} 1 & 1 + \varepsilon \leq |z| \leq \sqrt{e} - \varepsilon; \\ 0 & |z| \leq 1 \text{ or } |z| \geq \sqrt{e}. \end{cases}$$

We note that one can construct such a φ so that it satisfies $\mathfrak{D}(\varphi) = O(\varepsilon^{-1})$ and $\omega(\varphi, t) = O(\varepsilon^{-1}t)$. This implies

$$\begin{aligned} \mathbb{P}_{F(0)} [n_{F_C}(\{r(1+\varepsilon) \leq |z| \leq \sqrt{e}r(1-\varepsilon)\}) > r^\gamma] &\leq \mathbb{P}_{F(0)} [n_{F_C}(\varphi; r) > r^\gamma] \\ &\leq \exp\left(-\frac{C}{\mathfrak{D}(\varphi)} r^{2\gamma}\right) \\ &\leq \exp(-C\varepsilon r^{2\gamma}), \end{aligned}$$

provided $r^\gamma > C_5 \sqrt{\mathfrak{D}(\varphi)} r \log r$ (which is satisfied if $\gamma \in \left(1 + \frac{1}{2} \log \frac{1}{\varepsilon} (\log r)^{-1}, 2\right]$, and r is sufficiently large).

7.4. Convergence of the counting measure - Proof of Theorem 1. The proof of Theorem 1 is straightforward and we include it for completeness. We will need the following result, we leave the simple proof to the reader.

Claim 12. Let X be a real random variable with finite mean, and $a \in \mathbb{R}$, $T \geq 0$. We have

$$\mathbb{E} |X - a| \leq T + \int_0^\infty \mathbb{P}[|X - a| > s + T] ds.$$

We write $X_\varphi = X_\varphi(r) = n_{F_C}(\varphi; r)|_{F(p)}$ for the random variable $n_{F_C}(\varphi; r)$ conditioned on the event $F(p)$. Applying the claim to $X = X_\varphi$, $a = r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w)$, $T = Cr \log^2 r$, and using Theorem 8, we get

$$\begin{aligned} \left| \mathbb{E}[X_\varphi] - r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w) \right| &\leq \mathbb{E} \left[\left| X_\varphi - r^2 \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w) \right| \right] \\ &\leq C \left(r \log^2 r + \sqrt{\mathfrak{D}(\varphi)} \right). \end{aligned}$$

Recall that $[Z_r^p]$ is the counting measure of the zeros of F_C on the event $F(p; r)$, and denote by $[\tilde{Z}_r^p] = \frac{1}{r^2} [Z_r^p] \left(\frac{\cdot}{r}\right)$ the scaled counting measure. In order to prove

$$[\tilde{Z}_r^p] \xrightarrow[r \rightarrow \infty]{v} \mu_{Z_p}^{\mathbb{C}} \quad \text{in distribution,}$$

we have to show that for every $\phi \in C_0(\mathbb{C})$ a continuous test function with compact support, the random variable

$$\int_{\mathbb{C}} \phi(w) d[\tilde{Z}_r^p](w) = \frac{1}{r^2} n_{F_C}(\phi; r)|_{F(p)} = r^{-2} X_\phi,$$

converges in distribution to $\int_{\mathbb{C}} \phi(w) d\mu_{Z_p}^{\mathbb{C}}(w)$ (since the limit is a constant, this is the same as convergence in probability). Suppose ϕ is supported on the disk $D(0, B)$, where $B \geq 1$, and let $\varphi \in C_0^2(\mathbb{C})$

be a smooth test function, supported on the disk $D(0, B+1)$, such that $|\varphi - \phi| \leq \delta$. In particular, we have

$$\begin{aligned} |X_\phi - X_\varphi| &\leq \delta \cdot n_{F_C}(B+1), \\ \left| \int_{\mathbb{C}} \phi(w) d\mu_{Z_p}^{\mathbb{C}}(w) - \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w) \right| &\leq C\delta B. \end{aligned}$$

By Theorem 6, we have

$$\mathbb{P}_{F(p)}[n_{F_C}(B+1) > C_{B,p}r^2] \leq \exp(-Cr^4),$$

for some constant $C_{B,p}$, depending only on B and p . Therefore,

$$\begin{aligned} &\mathbb{P}\left[\left|r^{-2}X_\phi - \int_{\mathbb{C}} \phi(w) d\mu_{Z_p}^{\mathbb{C}}(w)\right| > \varepsilon\right] \\ &\leq \mathbb{P}\left[\left|r^{-2}X_\varphi - \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w)\right| > \varepsilon - C\delta B\right] + \mathbb{P}[|X_\phi - X_\varphi| > \varepsilon r^2] \\ &\leq \mathbb{P}\left[\left|r^{-2}X_\varphi - \int_{\mathbb{C}} \varphi(w) d\mu_{Z_p}^{\mathbb{C}}(w)\right| > \varepsilon - C\delta B\right] + \mathbb{P}_{F(p)}\left[n_{F_C}(B+1) \geq \frac{\varepsilon}{\delta}r^2\right]. \end{aligned}$$

Choosing δ sufficiently small, depending on ε , ϕ , and p , and using Theorem 8, we find that

$$\mathbb{P}\left[\left|r^{-2}X_\phi - \int_{\mathbb{C}} \phi(w) d\mu_{Z_p}^{\mathbb{C}}(w)\right| > \varepsilon\right] \leq \exp(-C_{\varepsilon,\phi,p}r^4).$$

This concludes the proof of Theorem 9.

8. DISCUSSION

In this paper, we considered the GEF,

$$F_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}, \quad z \in \mathbb{C},$$

It is known that this is the only translation invariant zero set of a Gaussian entire function up to scaling ([HoKPV, Section 2.5]). We mention that there exist similar constructions for other domains with transitive groups of isometries (the hyperbolic plane, the Riemann sphere, the cylinder and the torus, see [HoKPV, Section 2.3] for some examples).

8.1. Asymptotic probability of large fluctuations in the number of zeros. The hyperbolic GAF is the following Gaussian Taylor series,

$$F_D(z) = \sum_{k=0}^{\infty} \xi_k z^k, \quad |z| < 1.$$

It is known that its zero set is invariant with respect to the isometries of the unit disk ([HoKPV, Section 2.3]). Peres and Virág [PeV] proved that this zero set is a determinantal point process (see [HoKPV, Chapter 4]); this is the only example of this type). Denote by $n_{F_D}(r)$ the number of zeros of F_D in $D(0, r)$ ($0 < r < 1$). Using the representation of $n_{F_D}(r)$ as a sum of independent Bernoulli random variables, they found the asymptotics of the hole probability, as $r \rightarrow 1$ (see [HoKPV, Corollary 5.1.8.]). More recent results about the hole probability for GAFs in the unit disk can be found in [BuNPS, SkK].

Let us denote by $n(r) = n_{F_C}(r)$ the number of zeros of the GEF inside the disk $D(0, r)$. As we mentioned in Section 4, the Edelman-Kostlan formula implies that $\mathbb{E}[n(r)] = r^2$. In the paper [SoT2],

Sodin and Tsirelson considered large fluctuations in the number of zeros of the GEF, and proved that for every $\delta \in (0, \frac{1}{4}]$,

$$\mathbb{P} [|n(r) - r^2| \geq \delta r^2] \leq \exp(-c(\delta) r^4), \quad \text{as } r \rightarrow \infty,$$

with some unspecified positive constant $c(\delta)$. In the case where the GEF has no zeros in the disk $D(0, r)$ (i.e. the ‘hole’ event) they showed $\mathbb{P}[n(r) = 0] \geq \exp(-Cr^4)$. In the paper [Ni1], the second author found that the logarithmic asymptotics of the hole probability are given by

$$\log \mathbb{P}[n(r) = 0] = -\frac{e^2}{4} r^4 + o(r^4), \quad r \rightarrow \infty.$$

This result was later generalized by the second author to include entire functions represented by Gaussian Taylor series with arbitrary coefficients (see [Ni3, Ni4]).

Let us denote by $[\mathcal{G}]$ the random counting measure of the infinite Ginibre ensemble. It is known that this process is a determinantal point process. In particular, for a compact set $K \subset \mathbb{C}$, the random variable $[\mathcal{G}](K)$ can be expressed as a sum of *independent* Bernoulli random variables ([HoKPV, Theorem 4.5.3 and Remark 4.5.4]). Shirai [Sh] proved the following result (corresponding to our Theorem 6):

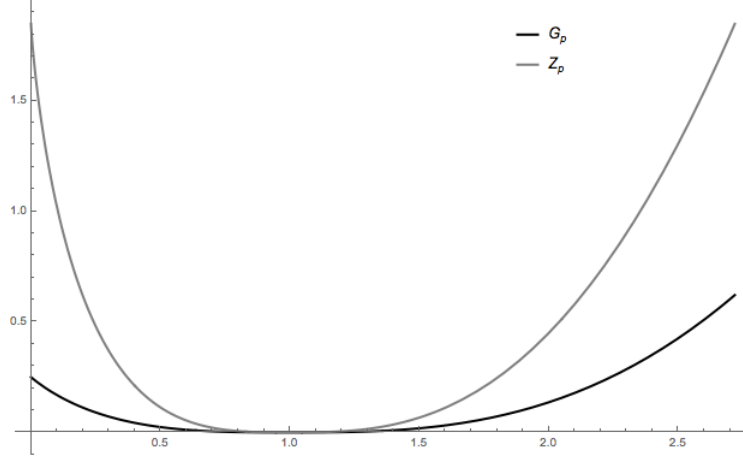
$$\mathbb{P} [[\mathcal{G}](D(0, r)) = [pr^2]] = \exp(-G_p \cdot r^4 + o(r^4)), \quad \text{as } r \rightarrow \infty,$$

where

$$G_p = \left| \int_1^p (1 - x + x \log x) \, dx \right|.$$

This provides a rigorous proof for some of the results of the paper [JaLM] (obtained for the finite Ginibre ensemble, in particular). The graphs of the constants G_p and Z_p are shown in Figure 8.1. Very recently, Adhikari and Reddy [AdR] found the asymptotics of the hole probability for non-circular domains (for both the finite and infinite Ginibre ensembles). We will consider this problem for the GEF in a future paper.

A problem similar to ours has been studied in the physical literature ([MaNSV]) in the one-dimensional setting. More precisely, consider Hermitian Gaussian random matrices (sampled from the GUE ensemble). Since the matrix is Hermitian, the eigenvalues are real, and the point process of eigenvalues is one-dimensional. On similar lines to our problem, we ask for the conditional distribution of the points given that there is a “gap” in the (macroscopic) interval $(-w\sqrt{N}, w\sqrt{N})$, where N is the dimension of the matrix and $w > 0$ is a fixed number. Considering a constrained variational problem somewhat analogous to our case, the authors are able to obtain a description of the minimizing measure. An important feature of this minimizing measure is that it has a density with respect to the Lebesgue measure (unlike the two-dimensional setting, where we find the appearance of a singular component in both the Ginibre and the GEF zero ensembles). Furthermore, there is no forbidden region, compared to our result in the case of the GEF zero ensemble.

Figure 8.1 - The constants G_p and Z_p , $p \in [0, e]$.

8.2. The Jancovici-Lebowitz-Manificat Law. In the paper [NaSV], Nazarov, Sodin, and Volberg studied a wider range of fluctuations in the random variable $n(r)$. For fixed $b > \frac{1}{2}$ and any $\varepsilon > 0$, they obtained the following result

$$-r^{\psi(b)+\varepsilon} \leq \log \mathbb{P} [|n(r) - r^2| > r^b] \leq -r^{\psi(b)-\varepsilon}, \quad r \geq r_0(b, \varepsilon),$$

where

$$\psi(b) = \begin{cases} 2b - 1 & \frac{1}{2} < b \leq 1; \\ 3b - 2 & 1 \leq b \leq 2; \\ 2b & b \geq 2. \end{cases}$$

Some of the cases were previously proved by Krishnapur in [Kr] (and also in [SoT2]), in particular he showed

$$\log \mathbb{P} [n(r) > r^b] = -\left(\frac{b}{2} - 1\right) (1 + o(1)) r^{2b} \log r, \quad b > 2, \quad r \rightarrow \infty.$$

These results are in agreement with a law discovered earlier by Jancovici, Lebowitz, and Manificat in their physical paper [JaLM]. This paper considered charge fluctuations of a one-component Coulomb system of particles of one sign embedded into a uniform background of the opposite sign (the finite Ginibre ensemble being a special case).

Our methods allows us to also consider smaller fluctuations in $n(r)$. For fixed constants a, b , with $a > 0$ and $b \in (\frac{4}{3}, 2)$, we have

$$(8.1) \quad \mathbb{P} [n(r) = \lfloor r^2 - ar^b \rfloor], \mathbb{P} [n(r) = \lfloor r^2 + ar^b \rfloor] = \exp \left(-\frac{2a^3}{3} \cdot r^{3b-2} (1 + o(1)) \right), \quad r \rightarrow \infty.$$

We can actually obtain the lower bound for $b \in (1, 2)$. Previously, Krishnapur ([Kr]) found the lower bound

$$\mathbb{P} [n(r) = \lfloor r^2 + ar^b \rfloor] \geq \exp (-a^3 r^{3b-2} (1 + o(1))), \quad b \in (1, 2), \quad r \rightarrow \infty.$$

It is plausible the result (8.1) holds in the whole range $b \in (1, 2)$.

8.3. Large deviations and the modified weighted energy functional $I(\nu)$. Let us fix $\alpha \geq e$, and for $N \in \mathbb{N}$ write $L = \sqrt{\alpha^{-1}N}$. Consider the following polynomials with independent standard complex Gaussian coefficients ξ_k ,

$$P_{N,L}(z) = \sum_{k=0}^N \xi_k \frac{(Lz)^k}{\sqrt{k!}}, \quad z \in \mathbb{C}.$$

Denote by $\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{z_j}$ the empirical measure of the zeros of $P_{N,L}$. In [ZeZ], Zeitouni and Zeldich prove a large deviation principle (LDP) for the sequence of measures μ_N (and for more general Gaussian polynomials). For $\nu \in \mathcal{M}_1(\mathbb{C})$, a probability measure on \mathbb{C} , let $U_\nu(z)$, $\Sigma(\nu)$ be its logarithmic potential and logarithmic energy, respectively (see the notation section for the definitions). In addition, let us define the following functional

$$(8.2) \quad I(\nu) = I_\alpha(\nu) = 2 \sup_{w \in \mathbb{C}} \left\{ U_\nu(w) - \frac{|w|^2}{2\alpha} \right\} - \Sigma(\nu),$$

which in the terminology of large deviations is called the *rate function*. The LDP means that for a Borel subset $\mathcal{C} \subset \mathcal{M}_1(\mathbb{C})$, we have

$$(8.3) \quad - \inf_{\nu \in \mathcal{C}^\circ} I(\nu) + A_\alpha \leq \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbb{P}[\mu_N \in \mathcal{C}] \leq \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbb{P}[\mu_N \in \mathcal{C}] \leq - \inf_{\nu \in \bar{\mathcal{C}}} I(\nu) + A_\alpha,$$

where $\bar{\mathcal{C}}$ (resp. \mathcal{C}°) is the closure (resp. interior) of \mathcal{C} in the weak topology, and $A_\alpha = \frac{\log \alpha}{2} - \frac{3}{4}$.

Remark 11. This functional was introduced for the first time in the paper [ZeZ]. In the papers [BeG, BeZ, HiP] on large deviations for (Gaussian) random matrices, the following functional appears

$$J(\nu) = J_\alpha(\nu) = \int_{\mathbb{C}} \frac{|w|^2}{\alpha} d\nu(w) - \Sigma(\nu).$$

In potential theory, the functional $J(\nu)$ is known as the *weighted energy functional* (see [SaT]).

Let $n_{P_{N,L}}(D)$ be the number of zeros of $P_{N,L}$ in the unit disk. Combining the LDP with the results of Section 5, gives (for a fixed α)

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbb{P}[n_{P_{N,L}}(D) \leq pL^2] &= -Z_p, \quad p \in (0, 1), \\ \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \mathbb{P}[n_{P_{N,L}}(D) \geq pL^2] &= -Z_p, \quad p \in (1, \alpha), \end{aligned}$$

where H_p is the constant appearing in Theorem 4. In the case $p = 0$, the LDP can only give a non-trivial upper bound, since the set $\mathcal{F}_0 = \{\nu \in \mathcal{M}_1(\mathbb{C}) : \nu(D) = 0\}$ has empty interior. Theorem 6 can be seen as an effective version of the LDP for the zeros of the GEF (for these particular questions).

8.4. The conditional distribution of the zeros. In the context of large deviations theory, the convergence of the empirical measure to a limit measure under conditioning is called the *Gibbs conditioning principle* (see [DeZ1], [DeZ2, Section 7.3]). This limit measure is given by the minimizer of a rate function under the constraint. In our case, the measure $\mu_{Z_p}^{\mathbb{C}}$ is the limit of measures $\alpha \mu_{Z_p}^\alpha$ as $\alpha \rightarrow \infty$. Here the probability measures $\mu_{Z_p}^\alpha$ are the minimizers of the functional $I(\nu)$ in (8.2).

The paper [JaLM] describes in particular the limiting conditional distribution for the finite Ginibre ensemble (i.e. the minimizers of the functional $J_\alpha(\nu)$). One obtains the following limiting measures:

$$d\tilde{\mu}_{Z_p}^\alpha(z) = \begin{cases} \frac{1}{\alpha} \left[\mathbf{1}_{\{|z| \leq \sqrt{p}\}}(z) + \mathbf{1}_{\{1 \leq |z| \leq \sqrt{\alpha}\}}(z) \right] \cdot \frac{dm(z)}{\pi} + \frac{1-p}{\alpha} dm_{\{|z|=1\}} & p \in [0, 1); \\ \frac{1}{\alpha} \left[\mathbf{1}_{\{|z| \leq 1\}}(z) + \mathbf{1}_{\{\sqrt{p} \leq |z| \leq \sqrt{\alpha}\}}(z) \right] \cdot \frac{dm(z)}{\pi} + \frac{p-1}{\alpha} dm_{\{|z|=1\}} & p \in (1, \alpha). \end{cases}$$

We see there is no additional “forbidden region” for the eigenvalues.

8.4.1. Simulation of the conditional distribution. It is possible to simulate the conditional distribution of the zeros (say on the hole event) using a modified Metropolis-Hastings algorithm [LaB], which takes into account the constraint. However, it seems like this method is only efficient in practice for a few hundreds of zeros (or eigenvalues in the case of the Ginibre ensemble). To produce the figures in Section 1 we used two different methods. Figure 1.1 (zeros conditioned on a hole) is generated using the ideas of the proof of the lower bound of Theorem 6 in Section 6. We generate random variables (which are no longer standard complex Gaussians), such that the GEF will have no zeros inside the disk of radius $r = 13$. Figure 1.2 is generated by simply moving the eigenvalues of a large random Ginibre matrix from the disk $\{|z| < 13\}$ to the boundary.

8.5. Large deviations for linear statistics. Let $\varphi \in C_0^2(\mathbb{C})$ be an arbitrary compactly supported test function. The following result is known as Offord’s estimate (see [HoKPV, Theorem 7.1.1.], [Sod]),

$$\begin{aligned} \mathbb{P} \left[\left| n_{F_c}(\varphi; r) - \frac{r^2}{\pi} \int_{\mathbb{C}} \varphi(w) dm(w) \right| \geq \lambda \right] &= \mathbb{P} \left[\left| \int_{\mathbb{C}} \varphi\left(\frac{w}{r}\right) dn_{F_c}(w) - \frac{1}{\pi} \int_{\mathbb{C}} \varphi\left(\frac{w}{r}\right) dm(w) \right| \geq \lambda \right] \\ &\leq 3 \exp \left(\frac{-\pi \lambda}{\|\Delta \varphi\|_{L^1(m)}} \right), \quad \lambda > 0, \end{aligned}$$

where we used $\|\Delta \varphi(\frac{\cdot}{r})\|_{L^1(m)} = \|\Delta \varphi\|_{L^1(m)}$. We mention that this bound is valid in general for Gaussian analytic functions. The following bound can be derived from our proof of Theorem 8,

$$\mathbb{P} \left[\left| n_{F_c}(\varphi; r) - \frac{r^2}{\pi} \int_{\mathbb{C}} \varphi(w) dm(w) \right| \geq \lambda \right] \leq \exp \left(-\frac{C}{\|\nabla \varphi\|_{L^2(m)}^2} \cdot \lambda^2 + O(r^2 \log^2 r) \right), \quad \lambda > 0.$$

APPENDIX A. THE JOINT DISTRIBUTION OF THE ZEROS

Let $L > 0$ and $N \in \mathbb{N}^+$. We want to find the joint probability density of the zeros of the polynomial

$$P(z) = P_{N,L}(z) = \sum_{k=0}^N \xi_k \frac{(Lz)^k}{\sqrt{k!}},$$

where ξ_k are i.i.d. standard complex Gaussians. This requires a change of variables, from the coefficients to the zeros (cf. the more general [ZeZ, Proposition 3]). We use the fact that the Jacobian determinant of this transformation can be expressed in a simple way in terms of the zeros.

Lemma 11. *Let $\underline{z} = (z_1, \dots, z_N)$ be the zeros of $P_{N,L}(z)$ in uniform random order. The joint distribution of \underline{z} , w.r.t. Lebesgue measure on \mathbb{C}^N , is given by*

$$f(\underline{z}) = f(z_1, \dots, z_N) = A_L^N |\Delta(\underline{z})|^2 \left(\int_{\mathbb{C}} \prod_{j=1}^N |w - z_j|^2 d\mu_L(w) \right)^{-(N+1)},$$

where

$$A_L^N = \frac{N! \cdot \prod_{j=1}^N j!}{\pi^N L^{N(N+1)}} = \exp \left(\frac{1}{2} N^2 \log \left(\frac{N}{L^2} \right) - \frac{3}{4} N^2 + O(N(\log N + \log L)) \right).$$

Remark 12. Recall that $|\Delta(\underline{z})|^2 = \prod_{j \neq k} |z_j - z_k|$ and $d\mu_L(w) = \frac{L^2}{\pi} e^{-L^2|w|^2} dm(w)$, where m is Lebesgue measure on \mathbb{C} .

Proof. Let $\underline{\xi} = (\xi_0, \dots, \xi_N)$. The joint density of $\underline{\xi}$ w.r.t. Lebesgue measure on \mathbb{C}^{N+1} is given by

$$(A.1) \quad g(\underline{\xi}) = \frac{1}{\pi^{N+1}} \exp \left(- \sum_{k=0}^N |\xi_k|^2 \right), \quad \underline{\xi} \in \mathbb{C}^{N+1}.$$

We now define the monic polynomials corresponding to $P_{N,L}(z)$,

$$q_{\underline{z}}(z) = \frac{P_{N,L}(z)}{\xi_N \cdot \frac{L^N}{\sqrt{N!}}} = z^N + b_{N-1} z^{N-1} + \dots + b_0 = \prod_{j=1}^N (z - z_j),$$

where

$$b_k = \frac{\xi_k}{\xi_N} \cdot \frac{\sqrt{N!}}{\sqrt{k!}} \cdot L^{k-N}, \quad k \in \{0, \dots, N-1\}.$$

The Jacobian of the map $T_1 : \underline{z} \mapsto \underline{b}$ that takes the zeros of the polynomial to the coefficients is given by $|\Delta(\underline{z})|^2$ (see for example [HoKPV, Lemma 1.1.1]). Clearly, the Jacobian of the (complex) linear map $T_2 : \underline{b} \mapsto \underline{\xi}$ is given by

$$\prod_{k=0}^{N-1} \frac{N!}{|\xi_N|^2 k! \cdot L^{2(N-k)}} = \frac{(N!)^{N+1}}{|\xi_N|^{2N} \prod_{k=1}^N k! \cdot L^{N(N+1)}} \stackrel{\text{def}}{=} |\xi_N|^{-2N} \cdot A'.$$

Therefore, after doing the change of variables from (ξ_0, \dots, ξ_N) to (\underline{z}, ξ_N) , and using Claim 13, we arrive at the joint density,

$$g'(\underline{z}, \xi_N) = \frac{1}{\pi^{N+1}} \cdot \frac{|\xi_N|^{2N}}{A'} \cdot |\Delta(\underline{z})|^2 \cdot \exp \left(- |\xi_N|^2 \frac{L^{2N}}{N!} \cdot \int_{\mathbb{C}} |q_{\underline{z}}(w)|^2 d\mu_L(w) \right).$$

We now integrate out ξ_N , and use the fact

$$\frac{1}{\pi} \int_{\mathbb{C}} |w|^{2N} e^{-B|w|^2} dm(w) = N! \cdot B^{-(N+1)},$$

to get

$$\begin{aligned} f(\underline{z}) &= \frac{1}{\pi^N} \cdot \frac{1}{A'} \cdot N! \left(\frac{N!}{L^{2N}} \right)^{N+1} |\Delta(\underline{z})|^2 \left[\int_{\mathbb{C}} |q_{\underline{z}}(w)|^2 d\mu_L(w) \right]^{-(N+1)} \\ &= \frac{N! \cdot \prod_{k=1}^N k!}{\pi^N L^{N(N+1)}} |\Delta(\underline{z})|^2 \left[\int_{\mathbb{C}} |q_{\underline{z}}(w)|^2 d\mu_L(w) \right]^{-(N+1)}. \end{aligned}$$

Stirling's approximation for the factorial shows that

$$N! \cdot \prod_{k=1}^N k! = \frac{N^2 \log N}{2} - \frac{3}{4} N^2 + O(N \log N).$$

□

Claim 13. Let $L > 0$ and $k \in \mathbb{N}$. We have

$$(A.2) \quad \int_{\mathbb{C}} |w|^{2k} d\mu_L(w) = \frac{k!}{L^{2k}}.$$

In addition, for $N \in \mathbb{N}^+$ let $P_{N,L}(z) = \sum_{k=0}^N \xi_k \frac{(Lz)^k}{\sqrt{k!}}$, and let $q_{\underline{z}}(z)$ be the corresponding monic polynomial. Then,

$$\int_{\mathbb{C}} |P_{N,L}(w)|^2 d\mu_L(w) = \sum_{k=0}^N |\xi_k|^2,$$

and thus

$$\int_{\mathbb{C}} |q_{\underline{z}}(w)|^2 d\mu_L(w) = \left(|\xi_N|^2 \frac{L^{2N}}{N!} \right)^{-1} \cdot \sum_{k=0}^N |\xi_k|^2.$$

Proof. Notice that for any $k \in \mathbb{N}$,

$$\begin{aligned} \int_{\mathbb{C}} |w|^{2k} d\mu_L(w) &= \frac{L^2}{\pi} \int_{\mathbb{C}} |w|^{2k} e^{-L^2|w|^2} dm(w) = 2L^2 \int_0^\infty t^{2k+1} e^{-L^2 t^2} dt \\ &= L^2 \int_0^\infty s^k e^{-L^2 s} ds = \frac{k!}{L^{2k}}. \end{aligned}$$

Therefore, using the orthogonality of z^j and \bar{z}^k w.r.t. the measure μ_L , we have

$$\int_{\mathbb{C}} |P_{N,L}(w)|^2 d\mu_L(w) = \sum_{k=0}^N |\xi_k|^2 \frac{L^{2k}}{k!} \cdot \int_{\mathbb{C}} |w|^{2k} d\mu_L(w) = \sum_{k=0}^N |\xi_k|^2.$$

□

The following estimate is sometimes called the Bernstein-Markov property of the measure μ_L (cf. [ZeZ, pg. 3939]).

Lemma 12. Let $L > 0$ and let h be a polynomial of degree N . We have

$$\sup_{w \in \mathbb{C}} \left\{ |h(w)|^2 e^{-L^2|w|^2} \right\} \leq \int_{\mathbb{C}} |h(z)|^2 d\mu_L(z).$$

Proof. Introduce the reproducing kernel (with respect to μ_L)

$$\Pi_{N,L}(w, z) = \sum_{k=0}^N \frac{L^{2k}}{k!} (w\bar{z})^k, \quad z, w \in \mathbb{C}.$$

It has the following basic properties:

1. $h(w) = \int_{\mathbb{C}} \Pi_{N,L}(w, z) h(z) d\mu_L(z),$
2. $\Pi_{N,L}(w, w) = \int_{\mathbb{C}} |\Pi_{N,L}(w, z)|^2 d\mu_L(z),$
3. $\Pi_{N,L}(w, w) \leq e^{L^2|w|^2}.$

The first two properties follow from (A.2) and Property 3 is clear from the definition. Applying the Cauchy-Schwarz inequality, we find

$$\begin{aligned} |h(w)|^2 &\leq \left(\int_{\mathbb{C}} |h(z)|^2 d\mu_L(z) \right) \left(\int_{\mathbb{C}} |\Pi_{N,L}(w, z)|^2 d\mu_L(z) \right) \\ &\leq \left(\int_{\mathbb{C}} |h(z)|^2 d\mu_L(z) \right) e^{L^2|w|^2}, \end{aligned}$$

where we used the fact μ_L is a probability measure. \square

APPENDIX B. SOME BACKGROUND ON LOGARITHMIC POTENTIAL THEORY

All the required background on weighted logarithmic potential theory can be found in the book [SaT], notice that we use here the opposite sign convention for the logarithmic potential of a measure.

Let $\nu \in \mathcal{M}_1(\mathbb{C})$ be a probability measure. Consider the following weighted energy functional

$$J(\nu) = J_\alpha(\nu) = \int_{\mathbb{C}} \frac{|w|^2}{\alpha} d\nu(w) - \Sigma(\nu),$$

where $\alpha > 0$ is a parameter. We recall that the logarithmic potential and logarithmic energy of ν are given by:

$$U_\nu(z) = \int_{\mathbb{C}} \log|z - w| d\nu(w), \quad \Sigma(\nu) = \int_{\mathbb{C}} U_\nu(z) d\nu(z) = \int_{\mathbb{C}^2} \log|z - w| d\nu(z) d\nu(w).$$

The logarithmic energy $\Sigma(\nu)$ is an upper semi-continuous and strictly concave functional, on measures with finite logarithmic energy and compact support ([HiP, Proposition 2.2]). Since $\int_{\mathbb{C}} |w|^2 d\nu(w)$ is a continuous linear functional, it follows that the functional $J(\nu)$ is lower semi-continuous and strictly convex.

It is known (see [SaT, Example IV.6.2], but notice the different scaling) that the global minimizer of this functional is the uniform measure on the disk $D(0, \sqrt{\alpha})$ which we denote by μ_{eq}^α . This measure is sometimes called the equilibrium or extremal measure. An easy calculation shows

$$U_{\mu_{\text{eq}}^\alpha}(z) = \begin{cases} \frac{|z|^2}{2\alpha} + \frac{\log \alpha}{2} - \frac{1}{2} & |z| \leq \sqrt{\alpha}; \\ \log|z| & |z| \geq \sqrt{\alpha}, \end{cases} \quad \Sigma(\mu_{\text{eq}}^\alpha) = \frac{\log \alpha}{2} - \frac{1}{4},$$

and

$$F_\alpha \stackrel{\text{def}}{=} \int_{\mathbb{C}} \frac{|w|^2}{2\alpha} d\mu_{\text{eq}}^\alpha(w) - \Sigma(\mu_{\text{eq}}^\alpha) = \frac{1}{4} - \left(\frac{\log \alpha}{2} - \frac{1}{4} \right) = \frac{1}{2} - \frac{\log \alpha}{2}.$$

Let \mathcal{H} be the set of all subharmonic functions $g(z)$ on \mathbb{C} that are harmonic for large $|z|$, and $g(z) - \log|z|$ is bounded from above near ∞ . By Theorem I.4.1 in [SaT], we have

$$U_{\mu_{\text{eq}}^\alpha}(z) + F_\alpha = \frac{|z|^2}{2\alpha} = \sup \left\{ g(z) : g \in \mathcal{H} \text{ and } g(w) \leq \frac{|w|^2}{2\alpha}, \forall |w| \leq \sqrt{\alpha} \right\}.$$

We summarize the implications in the following claim.

Claim 14. Let $\nu \in \mathcal{M}_1(\mathbb{C})$ be a probability measure with compact support, and define

$$B(\nu) = 2 \sup_{w \in \mathbb{C}} \left\{ U_\nu(w) - \frac{|w|^2}{2\alpha} \right\}.$$

We have

$$B(\nu) = B_\alpha(\nu) \stackrel{\text{def}}{=} 2 \sup_{|w| \leq \sqrt{\alpha}} \left\{ U_\nu(w) - \frac{|w|^2}{2\alpha} \right\}.$$

Proof. Notice $U_\nu(z)$ is a subharmonic function, that is harmonic for large $|z|$, and $U_\nu(w) - \log|z|$ is bounded from above near ∞ . Since,

$$U_\nu(z) - \frac{B_\alpha(\nu)}{2} \leq \frac{|z|^2}{2\alpha}, \quad z \in D(0, \sqrt{\alpha}),$$

we have

$$U_\nu(z) - \frac{B_\alpha(\nu)}{2} \leq U_{\mu_{\text{eq}}^\alpha}(z) + F_\alpha = \frac{|z|^2}{2\alpha}, \quad z \in \mathbb{C},$$

which implies $B(\nu) \leq B_\alpha(\nu)$. □

REFERENCES

- [AdR] K. Adhikari, N. K. Reddy, Hole probabilities for finite and infinite Ginibre ensembles. arXiv:1604.08363 [math.PR].
- [ArSZ] S. Armstrong, S. Serfaty, O. Zeitouni, Remarks on a constrained optimization problem for the Ginibre ensemble. *Potential Anal.* 41 (2014), no. 3, 945–958.
- [BeG] G. Ben Arous, A. Guionnet, Large deviations for Wigner’s law and Voiculescu’s non-commutative entropy. *Probab. Theory Related Fields* 108 (1997), no. 4, 517–542.
- [BeZ] G. Ben Arous, O. Zeitouni, Large deviations from the circular law. *ESAIM Probab. Statist.* 2 (1998), 123–134.
- [BoBL1] E. Bogomolny, O. Bohigas, P. Leboeuf, Quantum chaotic dynamics and random polynomials, *Journal of Statistical Physics*, 1996, Volume 85, Issue 5, pp 639–679. arXiv: chao-dyn/9604001.
- [BoBL2] E. Bogomolny, O. Bohigas, P. Leboeuf, Distribution of roots of random polynomials. *Phys. Rev. Lett.* 68 (1992), 2726–2729.
- [Bu] J. Buckley, Fluctuations in the zero set of the hyperbolic Gaussian analytic function. *Int. Math. Res. Not.* IMRN 2015, no. 6, 1666–1687. arXiv:1307.6674 [math.CV].
- [BuNPS] J. Buckley, A. Nishry, R. Peled, M. Sodin, Hole probability for zeroes of Gaussian Taylor series with finite radii of convergence. Submitted for publication. arXiv:1602.03076 [math.CV]
- [CoGHJK] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth, On the Lambert W Function. *Adv. Comput. Math.* 5 (1996), no. 4, 329–359.
- [DeZ1] A. Dembo, O. Zeitouni, Refinements of the Gibbs conditioning principle. *Probab. Theory Related Fields* 104 (1996), no. 1, 1–14.
- [DeZ2] A. Dembo, O. Zeitouni, Large deviations techniques and applications. Corrected reprint of the second (1998) edition. *Stochastic Modelling and Applied Probability*, 38. Springer-Verlag, Berlin, 2010.
- [FoH] P. J. Forrester, G. Honner, Exact statistical properties of the zeros of complex random polynomials. *J. Phys. A* 32 (1999), 2961–2981. arXiv:cond-mat/9812388 [cond-mat.stat-mech].
- [GhKP] S. Ghosh, M. Krishnapur, Y. Peres, Continuum Percolation for Gaussian zeroes and Ginibre eigenvalues, To appear in *Annals of Probability*. arXiv:1211.2514 [math.PR].
- [GhP] S. Ghosh, Y. Peres, Rigidity and Tolerance in point processes: Gaussian zeroes and Ginibre eigenvalues. To appear in *Duke Mathematical Journal*. arXiv:1211.2381 [math.PR].
- [GhZ] S. Ghosh, O. Zeitouni. Large deviations for zeros of random polynomials with i.i.d. exponential coefficients. To appear in *Int. Math. Res. Not.* arXiv:1312.6195 [math.PR]
- [Gi] J. Ginibre, Statistical Ensembles of Complex, Quaternion, and Real Matrices, *J. Math. Phys.* 6 (1965), 440–449.
- [HoKPV] J. B. Hough, M. Krishnapur, Y. Peres, B. Virág, Zeros of Gaussian Analytic Functions and Determinantal Point Processes. American Mathematical Society, 2010, (University lecture series ; v. 51).
- [Han] J. H. Hannay, Chaotic analytic zero points: exact statistics for those of a random spin state. *J. Phys. A* 29 (1996), L101–L105; J. H. Hannay, The chaotic analytic function. *J. Phys. A* 31 (1998), L755–L761.
- [HiP] F. Hiai, D. Petz, Maximizing free entropy. *Acta Math. Hungar.* 80 (1998), no. 4, 335–356.
- [JaLM] B. Jancovici, J. L. Lebowitz, G. Manificat, Large charge fluctuations in classical Coulomb systems. *J. Statist. Phys.* 72 (1993), 773–787.

- [Kr] M. Krishnapur, Overcrowding estimates for zeroes of planar and hyperbolic Gaussian analytic functions. *J. Stat. Phys.* 124 (2006), no. 6, 1399–1423.
- [La] N. S. Landkof, *Foundations of Modern Potential Theory*, Nauka, Moscow, 1966 (Russian). English transl.: Springer, New York–Heidelberg, 1972.
- [LaB] D. P. Landau, K. Binder, *A Guide to Monte Carlo Simulations in Statistical Physics*, Cambridge University Press; 4th edition, 2014.
- [Le] J. Lebowitz, Charge fluctuations in Coulomb systems, *Physical Review A*, 27, 1491–1494 (1983).
- [MaNSV] S. Majumdar, C. Nadal, A. Scardicchio, P. Vivo, How many eigenvalues of a Gaussian random matrix are positive?, *Physical Review E* 83.4 (2011).
- [NaS1] F. Nazarov, M. Sodin, Random complex zeroes and random nodal lines, *Proceedings of the International Congress of Mathematicians. Volume III*, 1450–1484, Hindustan Book Agency, New Delhi, 2010.
- [NaS2] F. Nazarov, M. Sodin, Correlation functions for random complex zeroes: strong clustering and local universality. *Comm. Math. Phys.* 310 (2012), no. 1, 75–98. arXiv:1005.4113 [math-ph].
- [NaS3] F. Nazarov, M. Sodin, What is... a Gaussian entire function? *Notices Amer. Math. Soc.* 57 (2010), no. 3, 375–377.
- [NaSV] F. Nazarov, M. Sodin, A. Volberg, The Jancovici - Lebowitz - Manificat law for large fluctuations of random complex zeroes, *Comm. Math. Phys.* 284 (2008), 833–865. arXiv:0707.3863 [math.PR].
- [Ni1] A. Nishry, Asymptotics of the hole probability for zeros of random entire functions. *Int. Math. Res. Not. IMRN* 2010, no. 15, 2925–2946. arXiv:0903.4970 [math.CV].
- [Ni3] A. Nishry, Hole probability for entire functions represented by Gaussian Taylor series. *J. Anal. Math.* 118 (2012), no. 2, 493–507. arXiv:1105.5734 [math.CV].
- [Ni4] A. Nishry, *Topics in the Value Distribution of Random Analytic Functions* (Ph.D. Thesis - Tel Aviv University), arXiv:1310.7542.
- [PeV] Y. Peres and B. Virág, Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process, *Acta Math.* 194 (2005), 1–35. arXiv:math/0310297 [math.PR].
- [PeH] D. Petz, F. Hiai, Logarithmic energy as an entropy functional. *Advances in differential equations and mathematical physics* (Atlanta, GA, 1997), 205–221, *Contemp. Math.*, 217, Amer. Math. Soc., Providence, RI, 1998.
- [Pr] I. E. Pritsker, Equidistribution of points via energy. *Ark. Mat.* 49 (2011), no. 1, 149–173.
- [Ro] P. C. Rosenbloom, Perturbation of the zeros of analytic functions. I. *J. Approximation Theory* 2 1969 111–126.
- [SaT] E. B. Saff, V. Totik, *Logarithmic potentials with external fields. Appendix B by Thomas Bloom. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 316. Springer-Verlag, Berlin, 1997. xvi+505 pp. ISBN: 3-540-57078-0
- [Sh] T. Shirai, Large deviations for the fermion point process associated with the exponential kernel. *J. Stat. Phys.* 123 (2006), no. 3, 615–629.
- [SkK] O. Skaskiv, A. Kuryliak, The probability of absence of zeros in the disc for some random analytic functions, *Math. Bull. Shevchenko Sci. Soc.*, 8 (2011), 335–352.
- [Sod] M. Sodin, Zeroes of Gaussian analytic functions. *Math. Res. Lett.* 7 (2000), no. 4, 371–381. arXiv:math/0007030 [math.CV].
- [Sos] A. Soshnikov, Determinantal random point fields, *Uspekhi Mat. Nauk* 55, (2000), no. 5(335), 107–160; translation in *Russian Math. Surveys* 55 (2000), no. 5, 923–975
- [SoT1] M. Sodin and B. Tsirelson, Random complex zeroes. I. Asymptotic normality, *Israel J. Math.* 144 (2004), 125–149. arXiv:math/0210090 [math.CV].
- [SoT2] M. Sodin, B. Tsirelson, Random Complex zeroes. III. Decay of the hole probability. *Israel J. Math.* 147 (2005), 371–379. arXiv:math/0312258 [math.CV].
- [Wi1] E. Wigner, On the Interaction of Electrons in Metals, *Phys. Rev.* 46, 1002. December 1934.
- [Wi2] E. Wigner, On the distribution of the roots of certain symmetric matrices, *Annals of Mathematics*, 67, pp 325–327 (1968).
- [ZeZ] O. Zeitouni, S. Zelditch, Large deviations of empirical measures of zeros of random polynomials. *Int. Math. Res. Not. IMRN* 2010, no. 20, 3935–3992. arXiv:0904.4271 [math.PR].

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